

Spherical Coordinates

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

$$\phi(r, \theta, \varphi) = R(r) \Theta(\theta) \Phi(\varphi)$$

$$r^2 \nabla^2 \phi = \Theta \Phi \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{R \Phi}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{R \Theta}{\sin^2 \theta} \frac{d^2 \Phi}{d\varphi^2} = 0$$

$$\frac{r^2 \sin^2 \theta}{\Phi} \nabla^2 \phi = \frac{\sin^2 \theta}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = 0$$

depends only on r and θ

$$= -\text{const}$$

depends only on φ

$$= \text{const}$$

take $\frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = -m^2$

$$\Rightarrow \boxed{\Phi = e^{\pm i m \varphi}}$$

m integer for 2π periodicity in φ

$$\Rightarrow \frac{\sin^2 \theta}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) = m^2$$

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0$$

depends only on r

$$= \text{const}$$

depends only on θ

$$= -\text{const}$$

call the const $\ell(\ell+1)$

For R

$$\frac{1}{r} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - \ell(\ell+1) = 0$$

Solutions are of the form
substitute in to verify

$$R(r) = a_\ell r^\ell + b_\ell r^{-(\ell+1)}$$

$$\begin{aligned} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) &= \frac{d}{dr} \left(r^2 (\ell a_\ell r^{\ell-1} - (\ell+1) b_\ell r^{-\ell-2}) \right) \\ &= \frac{d}{dr} \left(\ell a_\ell r^{\ell+1} - (\ell+1) b_\ell r^{-\ell} \right) \\ &= \ell(\ell+1) a_\ell r^\ell + \ell(\ell+1) b_\ell r^{-(\ell+1)} = \ell(\ell+1) R \end{aligned}$$

For Θ :

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = -\ell(\ell+1)$$

$$\text{let } x = \cos \theta$$

$$dx = -\sin \theta d\theta$$

$$d\theta = -\frac{dx}{\sin \theta}$$

above becomes

$$0 \leq \theta \leq \pi$$

solutions for $-1 \leq x \leq 1$

correspond to $\ell \geq 0$ integers

$$\frac{d}{dx} \left[(1-x^2) \frac{d\Theta}{dx} \right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2} \right] \Theta = 0$$

Called generalized Legendre Equation - solutions are
called the associated Legendre functions.

Ordinary Legendre polynomials are solutions
for $m=0$

For the special case $m=0$, i.e. the solution has azimuthal symmetry and Φ does not depend on the angle φ (i.e. rotational symmetry about $\hat{z}$ axis),

We want the solutions to

$$\frac{d}{dx} \left[(1-x^2) \frac{d\Phi}{dx} \right] + l(l+1) \Phi = 0$$

The solutions are known as the Legendre polynomials, $P_l(x)$.

They are given, for l integer, by

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx} \right)^l (x^2-1)^l \quad \text{Rodriguez's formula}$$

The lowest l polynomials are

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

In general, $P_l(x)$ is a polynomial of order l with only even powers if l is even, and only odd powers if l is odd. $\Rightarrow P_l(x) \begin{cases} \text{even in } x & \text{for } l \text{ even} \\ \text{odd in } x & \text{for } l \text{ odd} \end{cases}$

$P_l(x)$ is normalized so that $P_l(1) = 1$

Note: Legendre polynomials are only for integer $l \geq 0$.
What about solutions for non integer l ?

The $P_l(x)$ give one solution for each integer l .

But $P_l(x)$ are defined by a 2nd order differential equation - shouldn't there be a 2nd independent solution for each l ?

It turns out that these "2nd" solutions, as well as solutions for non integer l , all blow up at either $x = -1$ or $x = 1$, i.e. at $\theta = 0$ or $\theta = \pi$.

They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval $-1 \leq x \leq 1$.

$$\int_{-1}^1 dx P_l(x) P_m(x) = \int_0^\pi d\theta \sin\theta P_l(\cos\theta) P_m(\cos\theta) = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases}$$

$\Rightarrow$ we can expand any function $f(\theta)$, $0 \leq \theta \leq \pi$, as a linear combination of the $P_l(\cos\theta)$.

This is the reason they are useful for solving problems of Laplace's eqn with spherical boundary surfaces.

For $m \neq 0$, the solutions to (see Jackson 3.5)

$$\frac{d}{dx} \left[(1-x^2) \frac{d\Phi}{dx} \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] \Phi = 0$$

are the associated Legendre functions $P_l^m(x)$.

For $P_l^m(x)$ to be finite in interval $-1 \leq x \leq 1$

one again finds that l must be integer $l \geq 0$, and integer m must satisfy $|m| \leq l$, i.e. $m = -l, -(l-1), \dots, 0, \dots, (l-1), l$.

For each l and m there is only one such non divergent solution.

It is typical to combine the solutions $P_l^m(\cos\theta)$ to the θ -part of the equation with the $\Phi_m(\varphi) = e^{im\varphi}$ solutions to the φ -part of the equation to define the spherical harmonics

$$Y_{lm}(\theta, \varphi) \equiv \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}$$

The Y_{lm} are orthogonal

$$\int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{ll'} \delta_{mm'}$$

and are a complete set of basis functions for expanding any function $f(\theta, \varphi)$ defined on the surface of a sphere.

Examples with azimuthal symmetry $m=0$

General solution to $\nabla^2\phi=0$ can be written in form

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta)$$

determine the A_l and B_l from the boundary conditions of the particular problem.

- ① Suppose one is given $\phi(R, \theta) = \phi_0(\theta)$ on surface of sphere of radius R .

To find solution of $\nabla^2\phi=0$ inside sphere

ϕ should not diverge at origin $\Rightarrow B_l = 0$ for all l

$$\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$\Rightarrow \phi(R, \theta) = \phi_0(R) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

$$\begin{aligned} \Rightarrow \int_0^{\pi} d\theta \sin\theta \phi_0(\theta) P_m(\cos\theta) &= \sum_{l=0}^{\infty} A_l R^l \int_0^{\pi} d\theta \sin\theta P_l(\cos\theta) P_m(\cos\theta) \\ &= \sum_{l=0}^{\infty} A_l R^l \left(\frac{2}{2l+1} \right) \delta_{lm} \end{aligned}$$

$$= A_m R^m \frac{2}{2m+1}$$

$$A_m = \frac{2m+1}{2R^m} \int_0^{\pi} d\theta \sin\theta \phi_0(\theta) P_m(\cos\theta)$$

gives solution

To find solution of $\nabla^2 \phi = 0$ outside sphere

if require $\phi \rightarrow 0$ as $r \rightarrow \infty$, then $A_l = 0$ for all l

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$\phi(R, \theta) = \phi_0(\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)$$

gives
solution

$$B_m = \frac{2m+1}{2} R^{m+1} \int_0^\pi \sin \theta \phi_0(\theta) P_m(\cos \theta) d\theta$$

$$B_m = A_m R^{2m+1}$$

- (2) Suppose one is given surface charge density $\sigma(\theta)$ fixed on surface of sphere of radius R . What is ϕ inside and outside?

From previous example

$$\phi(r, \theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) & r < R \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) & r > R \end{cases}$$

boundary conditions at $r = R$ on surface

(i) ϕ continuous

$$\rightarrow \sum_{l=0}^{\infty} \left[A_l R^l - \frac{B_l}{R^{l+1}} \right] P_l(\cos \theta) = 0$$

If an expansion in Legendre polynomials vanishes for all θ , then each coefficient in the expansion must vanish

$$\Rightarrow A_l R^l = \frac{B_l}{R^{l+1}} \Rightarrow \boxed{B_l = A_l R^{2l+1}}$$

(ii) jump in electric field at σ

$$-\left. \frac{\partial \phi}{\partial r} \right|_{r=R}^{\text{out}} + \left. \frac{\partial \phi}{\partial r} \right|_{r=R}^{\text{in}} = 4\pi\sigma$$

$$\Rightarrow \sum_{l=0}^{\infty} \left[\frac{(l+1) B_l}{R^{l+2}} + l A_l R^{l-1} \right] P_l(\cos\theta) = 4\pi\sigma$$

$$\Rightarrow \sum_{l=0}^{\infty} \left[\frac{(l+1) A_l R^{2l+1}}{R^{l+2}} + l A_l R^{l-1} \right] P_l(\cos\theta)$$

$$\Rightarrow \sum_{l=0}^{\infty} (2l+1) R^{l-1} A_l P_l(\cos\theta) = 4\pi\sigma$$

$$(2m+1) R^{m-1} A_m \left(\frac{2}{2m+1} \right) = 4\pi \int_0^{\pi} d\theta \sin\theta \sigma(\theta) P_m(\cos\theta)$$

$$\boxed{A_m = \frac{4\pi}{2 R^{m-1}} \int_0^{\pi} d\theta \sin\theta \sigma(\theta) P_m(\cos\theta)}$$

Suppose $\sigma(\theta) = k \cos \theta$ what is ϕ ?

Note $\sigma(\theta) = k P_1(\cos \theta)$

hence only $A_1 \neq 0$ by orthogonality of $P_0(\cos \theta)$

$$A_1 = \frac{4\pi k}{2} \int_0^\pi \sin \theta P_1(\cos \theta) P_1(\cos \theta) d\theta$$
$$= \frac{4\pi k}{2} \left(\frac{2}{2+1} \right) = \frac{4\pi k}{3}$$

$$\Rightarrow \phi(r, \theta) = \begin{cases} \frac{4\pi}{3} k r \cos \theta & r < R \\ \frac{4\pi}{3} k \frac{R^3}{r^2} \cos \theta & r > R \end{cases}$$

we will see that potential outside the sphere is that of an ideal dipole with dipole moment

$$p = \frac{4\pi}{3} R^3 k$$

Inside the sphere, the potential $\phi = \frac{4\pi}{3} k z$ where $z = r \cos \theta$. The electric field inside the sphere is therefore the constant

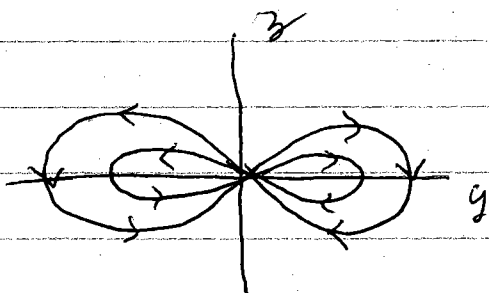
$$\vec{E} = -\vec{\nabla} \phi = -\frac{4\pi k}{3} \vec{z}$$

outside the sphere the field is

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

$$= \frac{8\pi k R^3}{3} \frac{\cos\theta}{r^3} \hat{r} + \frac{4\pi k R^3}{3} \frac{\sin\theta}{r^3} \hat{\theta}$$

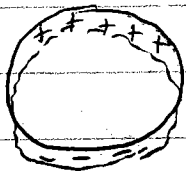
$$\vec{E} = \frac{4\pi R^3 k}{3} \frac{1}{r^3} [2\cos\theta \hat{r} + \sin\theta \hat{\theta}]$$



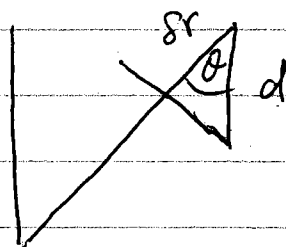
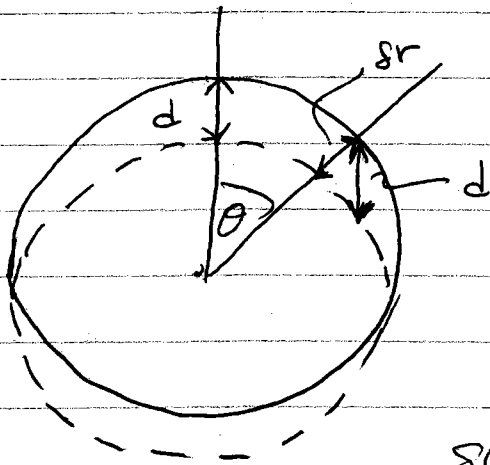
dipole field

Physical example with $\sigma(\theta) = k \cos \theta$

Two spheres of radii R , with equal but opposite uniform charge densities ρ and $-\rho$, displaced by small distance $d \ll R$



surface charge σ builds up due to displacement
This is a uniformly "polarized" sphere



$$d \cos \theta = r$$

surface charge is $\sigma(\theta) = \rho r$
 $= \rho d \cos \theta$

$$\boxed{\sigma(\theta) = \rho d \cos \theta}$$

total dipole moment is $(\rho d) \frac{4}{3} \pi R^3$

polarization = $\frac{\text{dipole moment}}{\text{volume}} = \rho d$

$\vec{E}$ field inside a uniformly polarized sphere is constant. $\vec{E} = -\rho d \frac{4\pi}{3}$

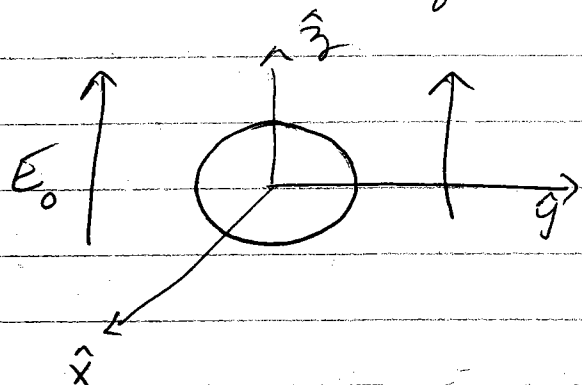
Grounded

③ Conducting sphere in uniform electric field $\vec{E} = E_0 \hat{z}$

as $r \rightarrow \infty$ far from sphere, $\vec{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$

boundary conditions $= -E_0 r \cos \theta$

$$\begin{cases} \phi(R, \theta) = 0 \\ \phi(r \rightarrow \infty, \theta) = -E_0 r \cos \theta \end{cases}$$



solution outside sphere has the form

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos \theta)$$

From boundary condition as $r \rightarrow \infty$ we have

$$A_l = 0 \quad \text{all } l \neq 1$$

$$A_1 = -E_0 \quad \text{since } P_1(\cos \theta) = \cos \theta$$

$$\phi(r, \theta) = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

From $\phi(R, \theta) = 0$ we have

$$0 = -E_0 R \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)$$

$$\Rightarrow B_l = 0 \quad \text{all } l \neq 1$$

$$\frac{B_1}{R^2} = E_0 R \Rightarrow B_1 = +E_0 R^3$$

So

$$\phi(r, \theta) = -E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta$$

1st term is just potential $-E_0 r \cos \theta$ of the uniform applied electric field.

2nd term is potential due to the induced surface charge on the surface - it is a dipole field

Induced charge density is

$$4\pi\sigma(\theta) = -\left. \frac{\partial\phi}{\partial r} \right|_{r=R} = E_0 \left(1 + \frac{2R^3}{R^3} \right) \cos \theta$$
$$= 3E_0 \cos \theta$$

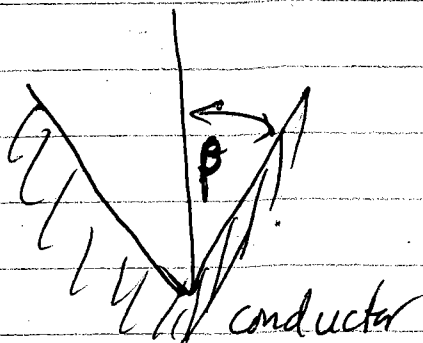
$$\sigma(\theta) = \frac{3}{4\pi} E_0 \cos \theta \quad \text{like uniformly polarized sphere} \quad k = \frac{3E_0}{4\pi}$$

from ② we know that the field inside the sphere due to this σ is just $-\frac{4}{3}\pi k \hat{z} = -\frac{4}{3}\pi \frac{3E_0}{4\pi} \hat{z}$

$= -E_0 \hat{z}$. This is just what is required so that the total field in the conducting sphere vanishes,

Can check that outside the sphere, $\vec{E} = -\vec{\nabla}\phi$ is normal to surface of sphere at $r=R$.

Behaviour of fields near conical hole or sharp tip



we now want to solve the $\nabla^2 \phi = 0$ with separation of variables, but now θ is restricted to range $0 \leq \theta \leq \beta$.

We still have azimuthal symmetry, but now, since we do not need solution to ϕ to be finite for all $\theta \in [0, \pi]$, but only $\theta \in (0, \beta)$, we have more solutions to the $\oplus$ equation, i.e. l does not have to be integer, - still need $l \geq 0$ to be finite at $\theta = 0$.

see Jackson sec. 3.4 for details.