Eigenfunction expansion for Green Functions

Suppose $\mathbb{D}$ is some linear differential operator, for example ∇^2.

Solutions to the equation

$$\mathbb{D}\,\psi(\vec{r}) = -4\pi f(\vec{r})$$

can be solved if one knows the Green function, which is the solution to the problem with a point source

$$\underset{\substack{\uparrow \\ \text{operates on } \vec{r}}}{\mathbb{D}} G(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,')$$

Then

$$\psi(\vec{r}) = \int d^3r'\, G(\vec{r},\vec{r}\,')\, f(\vec{r}\,') \qquad \text{is solution}$$

If we need to solve for ψ subject to certain boundary conditions, then we can always add to the Green function a $\phi(\vec{r})$ such that $\mathbb{D}\phi(\vec{r}) = 0$ in the desired region and then choose ϕ accordingly as we did for Dirichlet or Neumann b.c. for ∇^2.

One way to find $G(\vec{r},\vec{r}\,')$ is to find the eigenvalues and eigenfunctions of $\mathbb{D}$.

$$\mathbb{D}\,\underset{\substack{\uparrow \\ \text{eigenfunction}}}{\psi_n(\vec{r})} = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{\lambda_n}\,\psi_n(r)$$

Depending on the problem, the spectrum of eigenvalues might be discrete or might be continuous.

Note: When we solved Laplace's equation by separation of variables method, what we wound up doing was solving the eigenvalue problem for the (in spherical case) radial, θ, and φ pieces of the differential operator.

In many cases (you would have to prove this for the particular operator $\mathbb{D}$) the eigenfunctions $\psi_n(\vec{r})$ form an orthogonal and complete set of basis functions over the region of interest (ie in the volume in which we are seeking a solution)

orthogonal $\Rightarrow$ $\boxed{\int_V d^3r\ \psi_m^*(\vec{r})\,\psi_n(\vec{r}) = \delta_{m,n}}$

complete $\Rightarrow$ $f(\vec{r}) = \sum_n a_n \psi_n(\vec{r})$.

any function f can be expanded in a linear combination of the ψ_n.

The expansion coefficients a_n are obtained by

$$\int_V d^3r\ f(\vec{r})\,\psi_m^*(\vec{r}) = \sum_n a_n \int_V d^3r\ \psi_m^*(\vec{r})\,\psi_n(\vec{r}) = \sum_n a_n \delta_{m,n}$$

So $\boxed{a_m = \int_V d^3r\ f(\vec{r})\,\psi_m^*(\vec{r})}$ "Fourier" coefficient for basis ψ_n

In particular, the function $\delta(\vec{r}-\vec{r}\,')$ can be expanded as

$$\delta(\vec{r}-\vec{r}\,') = \sum_n a_n \psi_n(\vec{r})$$

where

$$a_n = \int_V d^3r\ \delta(\vec{r}-\vec{r}\,')\,\psi_n^*(\vec{r}) = \psi_n^*(\vec{r}\,') \qquad \text{assuming } \vec{r}\,' \in V$$

So we have $\boxed{\delta(\vec{r}-\vec{r}\,') = \sum_n \psi_n^*(\vec{r}\,')\psi_n(\vec{r})}$

Now we can solve for the Green function!

Expand $G(\vec{r},\vec{r}\,')$ as, $\underbrace{\text{a series}}_{\text{a function of } \vec{r}, \text{ in}}$ in $\psi_n(\vec{r})$

$$G(\vec{r},\vec{r}\,') = \sum_n a_n \psi_n(\vec{r})$$

Now use

$$\mathbb{D}\,G(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,')$$

since $\mathbb{D}$ is linear

$$\hookrightarrow \sum_n a_n \mathbb{D}\psi_n(\vec{r}) = \sum_n a_n \lambda_n \psi_n(\vec{r}) = -4\pi \sum_n \psi_n^*(\vec{r}\,')\psi_n(\vec{r})$$

$$\Rightarrow \sum_n \left[a_n\lambda_n + 4\pi\,\psi_n^*(\vec{r}\,') \right] \psi_n(\vec{r}) = 0$$

If a series in a set of basis functions vanishes then each coefficient in the series must vanish

$$\Rightarrow \quad a_n = \frac{-4\pi\,\psi_n^*(\vec{r}\,')}{\lambda_n}$$

$$\boxed{G(\vec{r},\vec{r}\,') = -4\pi \sum_n \left[\frac{\psi_n^*(\vec{r}\,')\psi_n(\vec{r})}{\lambda_n} \right]}$$

Example: ∇^2 in rectangular coordinate, V = all space

$$\nabla^2 \psi(\vec{r}) = \lambda \psi(\vec{r})$$

call the eigenvalues $\lambda = -k^2$

eigen functions are then $\psi \sim e^{i\vec{k}\cdot\vec{r}}$

check $\vec{\nabla}\psi = i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$

$$\nabla^2\psi = \vec{\nabla}\cdot(\vec{\nabla}\psi) = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2\psi$$

normalize ψ for orthogonality condition $\boxed{\psi_k(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}}$

$$\int d^3r\, \psi^*_{k'}(\vec{r})\, \psi_k(\vec{r}) = \int d^3r \frac{1}{(2\pi)} e^{-i\vec{k}'\cdot\vec{r}} e^{i\vec{k}\cdot\vec{r}}$$

$$= \int d^3r \frac{e^{i(\vec{k}-\vec{k}')\cdot\vec{r}}}{(2\pi)^3} = \delta(\vec{k}-\vec{k}')$$

$$\Rightarrow \quad G(\vec{r},\vec{r}') = -4\pi \int \frac{d^2k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}}{(-k^2)} = \int \frac{d^3k}{(2\pi)^3}\left(\frac{4\pi}{k^2}\right) e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

Now we already know that the Green function for this problem is

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|}$$

So from this we see that the Fourier transf of

$$\frac{1}{|\vec{r}-\vec{r}'|} \text{ is } \frac{4\pi}{k^2}$$

Example Green's function for Dirichlet problem inside rectangular box $x \in [0,a]$, $y \in [0,b]$, $z \in [0,c]$

We are looking for eigenfunction of

$$\nabla^2 \psi = \lambda \psi$$

with $\psi = 0$ on boundaries of the rectangular box.

Solutions are

$$\psi_{lmn} = \sqrt{\frac{8}{abc}} \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi z}{c}\right)$$

with eigenvalue $\lambda_{lmn} = -\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$, $lmn = 1, \ldots$

check normalization for yourselves!

$$G(\vec{r},\vec{r}') = -4\pi \sum_{lmn=1}^{\infty} \frac{8}{abc} \frac{\sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \times \sin\left(\frac{n\pi z}{c}\right) \sin\left(\frac{n\pi z'}{c}\right)}{-\pi^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)}$$

$$G(\vec{r},\vec{r}') = \frac{32}{\pi abc} \sum_{lmn=1}^{\infty} \frac{\sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{l\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right) \sin\left(\frac{n\pi z}{c}\right) \sin\left(\frac{n\pi z'}{c}\right)}{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)}$$

Note that in this case, $G(\vec{r},\vec{r}')$ is <u>NOT</u> a function of $\vec{r}-\vec{r}'$. The boundary breaks the translational invariance.

Magnetostatics

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad , \quad \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \dfrac{4\pi}{c}\vec{j} \end{cases}$$ Ampere's Law (statics only!)

$$\Rightarrow \quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j}$$

can write $\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

where by $\nabla^2\vec{A}$ we mean $(\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z}$

$\nabla^2\vec{A}$ only has a simple expression in Cartesian coords

Aside

If tried to write it in spherical coords, for example, one has

$$\begin{aligned}\nabla^2\vec{A} &= \nabla^2\left(A_r\hat{r} + A_\theta\hat{\theta} + A_\varphi\hat{\varphi}\right) \\ &= (\nabla^2 A_r)\hat{r} + A_r(\nabla^2\hat{r}) + (\nabla^2 A_\theta)\hat{\theta} + A_\theta(\nabla^2\hat{\theta}) \\ &\quad + (\nabla^2 A_\varphi)\hat{\varphi} + A_\varphi(\nabla^2\hat{\varphi})\end{aligned}$$

one must not forget to take the derivates of $\hat{r}, \hat{\theta}, \hat{\varphi}$ since they vary with position!

for example, $\hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$

one could compute $\nabla^2\hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla}\cdot\vec{A} = 0$, then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \boxed{-\nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}}$$ Poisson's equation!

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$.
But vector nature of eqn makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$ three equations for A_x, A_y, A_z !

for localized current sources $j(r) \to 0$ as $r \to \infty$

<u>Multipole expansion</u> - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use <u>vector spherical harmonics</u> - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2)^{1/2}} = \frac{1}{r}\frac{1}{\left[1 - \frac{2\vec{r}\cdot\vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2\right]^{1/2}}$$

do Taylor series to 1st order in $\left(\frac{r'}{r}\right)$ to get

$$\frac{1}{|\vec{r}-\vec{r}'|} \simeq \frac{1}{r}\left\{1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} + \cdots\right\} = \frac{1}{r} + \frac{\vec{r}\cdot\vec{r}'}{r^3} + \cdots$$