$$\vec{A}(\vec{r}) = \underbrace{\int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{r}}_{①} + \underbrace{\int \frac{d^3r'}{c} \vec{j}(\vec{r}') \frac{(\vec{r}\cdot\vec{r}')}{r^3}}_{②} + \cdots$$

Consider term ①

$$\int d^3r \, \vec{j}(\vec{r})$$

$$\int d^3r (\vec{j}\cdot\vec{\nabla})\vec{r} \qquad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

write $\displaystyle\int d^3r \, j_i(r) = \sum_{j=1}^{3} \int d^3r \, j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

$$= \sum_j \left[\oint_S da \, j_j r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right]$$

vanishes as $S \to \infty$ if $\vec{j}$ sufficiently localized ie $\vec{j}(\vec{r}) \to 0$ sufficiently fast as $r \to \infty$

vanishes in magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$

So $\int d^3r \, \vec{j}(\vec{r}) = 0$ in magnetostatics

monopole term vanishes

term (2)

$$\int d^3r \; \vec{J}(\vec{r})\,\vec{r} \qquad \text{tensor}$$

Consider $\int d^3r \; j_i r_j = \sum_k \int d^3r \; j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

$$= \sum_k \left\{ \oint_S da \; j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k}(j_k r_j) r_i \right\}$$

vanishes as $S \to \infty$ if $\vec{J}$ sufficiently localized

$$= -\sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

vanishes as $\vec{\nabla}\cdot\vec{J} = 0$ in magnetostatics $\qquad \frac{\partial r_j}{\partial r_k} = \delta_{jk}$

$$= -\int d^3r \; j_j r_i$$

So $\int d^3r \; j_i r_j = -\int d^3r \; j_j r_i$

$$= \frac{1}{2}\int d^3r \left(j_i r_j - j_j r_i \right)$$

So

$$\int d^3r' \; j_i(\vec{r}')(\vec{r}\cdot\vec{r}') = \sum_j r_j \int d^3r' \; j_i(\vec{r}') r_j'$$

$$= \sum_j \frac{1}{2}\int d^3r' \left(j_i r_j r_j' - r_j j_j r_i' \right)$$

$$= \frac{1}{2}\int d^3r' \left(j_i (\vec{r}\cdot\vec{r}') - r_i' (\vec{r}\cdot\vec{J}) \right)$$

use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{j}) = \vec{r}' (\vec{r} \cdot \vec{j}) - \vec{j} (\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \, \vec{j} \, (\vec{r} \cdot \vec{r}') = -\frac{1}{2} \vec{r} \times \left[\int d^3r' \, \vec{r}' \times \vec{j}(\vec{r}') \right]$$

define the magnetic dipole moment as

$$\boxed{\vec{m} = \frac{1}{2c} \int d^3r' \, \vec{r}' \times \vec{j}(\vec{r}')}$$

In magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}(\vec{r}) = -\frac{\vec{r} \times \vec{m}}{r^3} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3} \right)$$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{any two indices equal} \\ +1, & ijk \text{ even permutation of } 123 \\ -1, & ijk \text{ odd permutation } 123 \end{cases}$$

$$B_i = \varepsilon_{ijk} \, \partial_j \, \varepsilon_{klm} \, m_l \frac{r_m}{r^3}$$

$$= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \partial_j \, m_l \frac{r_m}{r^3}$$

$$= m_i \, \partial_j \left(\frac{r_j}{r^3} \right) - m_j \, \partial_j \left(\frac{r_i}{r^3} \right)$$

$$= m_i \left[-4\pi \delta(\vec{r}) \right] - m_j \left[\frac{\delta_{ij}}{r^3} - \frac{3 r_i}{r^4} \partial_j r \right]$$

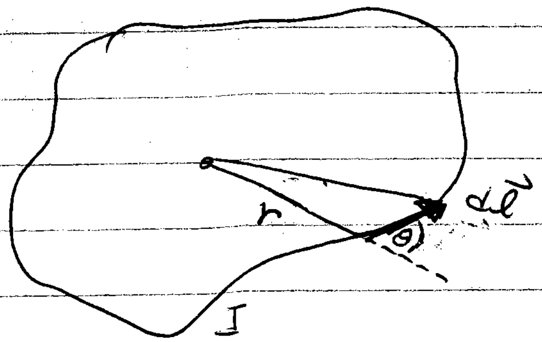
$$= \underbrace{0}_{\text{far from source}} - \frac{m_i}{r^3} + 3 \frac{r_i}{r^4} \frac{r_j}{r} m_j$$

$$\boxed{\vec{B} = \frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}}$$

same form as $\vec{E}$ from electric dipole $\vec{p}$

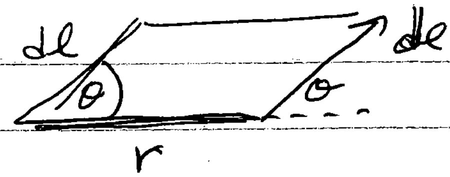
For a <u>current loop in a plane</u> (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r}\times\vec{j} = \frac{1}{2c} I \oint \vec{r}\times d\vec{l}$$



area of triangle is $\frac{1}{2} r\, dl \sin\theta$

$= \frac{1}{2} |\vec{r}\times d\vec{l}|$



area of rectangle is $r\, dl \sin\theta$

$$\Rightarrow \quad \vec{m} = \frac{1}{c} I (\text{area})\, \hat{n}$$

area of loop

outward normal (direction given by right hand rule with respect to direction of current) magnetic dipole moment $\vec{m}$ is independent of location of origin.

$$\vec{r}\,' = \vec{r} + \vec{d} \qquad \text{new coord}$$

$$\vec{m}\,' = \frac{1}{2c}\int d^3r' \left(\vec{r}\,' \times \vec{J}\right) = \frac{1}{2c}\int d^3r \; (\vec{r} + \vec{d}) \times \vec{J}$$

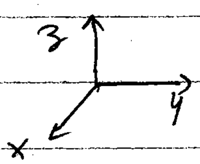
$$= \frac{1}{2c}\int d^3r \; \vec{r} \times \vec{J} + \frac{1}{2c}\vec{d} \times \left[\int d^3r \; \vec{J}\,\right]$$

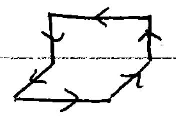
$$\vec{m}\,' = \vec{m} + 0 \quad \text{as} \quad \int d^3r \; \vec{J} = 0$$

for planar loop $\quad \vec{m} = \frac{I}{c} a \, \hat{n} \quad$ where a = area

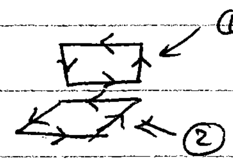
$\hat{n}$ = outward normal

Can also apply to get $\vec{m}$ for <u>piecewise planar loops</u>





= superposition of



$$\vec{m} = \vec{m}_1 + \vec{m}_2 \qquad \vec{m}_1 = \frac{I}{c} a_1 \hat{x}$$

$$\vec{m}_2 = \frac{I}{c} a_2 \hat{z}$$

$$\Rightarrow \quad \vec{m} = \frac{I}{c}\left(a_1 \hat{x} + a_2 \hat{z}\right)$$

Boundary value problems in magnetostatics
Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c}\vec{J}$$

and the fact that $\nabla^2 \vec{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar $-\nabla^2\phi = 4\pi\rho$ dont work so well for magnetostatics.

However, in situations where the current $\vec{J}$ is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential ϕ_M.

In regions where $\vec{J}=0$, ie not on the certain surfaces, we have $\vec{\nabla}\cdot\vec{B}=0$ and $\vec{\nabla}\times\vec{B}=0$. Since $\vec{\nabla}\times\vec{B}=0$ in these regions we can define a scalar potential ϕ_M such that

$$\vec{B} = -\vec{\nabla}\phi_M$$

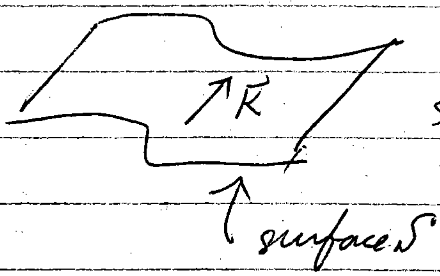
and then

$$\vec{\nabla}\cdot\vec{B} = -\nabla^2\phi_M = 0$$

We can solve for ϕ_M as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

Bounday Conditions at Sheet current

in magnetostatics $\vec\nabla\cdot\vec B = 0$, $\vec\nabla\times\vec B = \frac{4\pi}{c}\vec J$



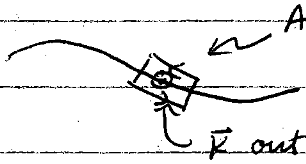
surface current $\vec K(\vec r)$ at pt $\vec r$ on surface S

Gaussian pillbox vol V $\quad \int_V d^3r\ \vec\nabla\cdot\vec B = 0$

top + bottom area of pill box is da
width of pill box $\to 0$

$$\Rightarrow \int_V d^3r\ \vec\nabla\cdot\vec B = \oint_S da\ \hat n\cdot\vec B = da(\vec B_{above} - \vec B_{below})\cdot\hat n = 0$$

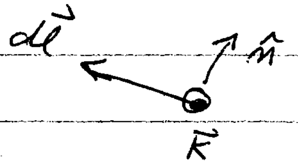
<u>normal component of $\vec B$ is continuous</u> $(\vec B_{above} - \vec B_{below})\cdot\hat n =$



Amperian loop C bounding area S

$$\int_S da\ \hat n\cdot(\vec\nabla\times\vec B) = \oint_C \vec B\cdot d\vec l = \frac{4\pi}{c} I_{enclosed}$$

let width of loop $\to 0$, top + bottom sides $d\vec l$



$$(\vec B_{above} - \vec B_{below})\cdot d\vec l = \frac{4\pi}{c}(\hat n\times d\vec l)\cdot\vec K$$
$$= \frac{4\pi}{c}(\vec K\times\hat n)\cdot d\vec l$$

$\hat n$ is outward normal

<u>tangential component of $\vec B$ has discontinuous jump $\frac{4\pi}{c}\vec K\times\hat n$</u>

Combine both results into

$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c} \vec{K} \times \hat{n}}$$

magnetic analog of $\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma \hat{n}$

In terms of magnetic vector potential ϕ_M

$$-\vec{\nabla}\phi_{M\,above} + \vec{\nabla}\phi_{M\,below} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note: ϕ_M is a calculational tool only
it does not have any direct physical significance as does the electrostatic ϕ.
Electrostatic ϕ is related to work done moving a charge $W_{12} = q[\phi(r_2) - \phi(r_1)]$
nothing similar for ϕ_M.
(in fact ~~magnetostatic~~ magnetic forces do no work!
$\vec{F} = q\, \vec{v} \times \vec{B}$
$\Rightarrow \vec{F} \cdot \vec{v} = \frac{dW}{dt} = 0$)

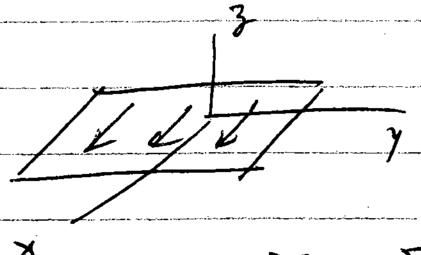
Note: ~~We cannot apply argument~~ like ~~$\phi(r) - \phi(r') = \int_{r'}^{r} \vec{E} \cdot d\vec{l}$~~
ϕ_M is not necessarily continuous at surface current
Cannot do similar to electrostatics and use
$$\phi_M(r_{above}) - \phi_M(r_{below}) = -\int_{r_{below}}^{r_{above}} \vec{B} \cdot d\vec{l}$$

Since ϕ_M is not defined on the current sheet itself, separating "above" from "below".

example

Flat infinite plane at $z=0$ with surface current

$$\vec{K} = K\hat{x}$$



$$z>0, \quad \nabla^2\Phi_M^> = 0 \Rightarrow \Phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$
$$z<0, \quad \nabla^2\Phi_M^< = 0 \Rightarrow \Phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z>0, \quad \vec{B}^> = -\vec{\nabla}\Phi_M^> = b_x^>\hat{x} + b_y^>\hat{y} + b_z^>\hat{z}$$
$$z<0, \quad \vec{B}^< = -\vec{\nabla}\Phi_M^< = b_x^<\hat{x} + b_y^<\hat{y} + b_z^<\hat{z}$$

at $z=0$ $\quad \vec{B}^> - \vec{B}^< = (b_x^> - b_x^<)\hat{x} + (b_y^> - b_y^<)\hat{y} + (b_z^> - b_z^<)\hat{z}$

$$= \frac{4\pi}{c}\vec{K}\times\hat{n} = \frac{4\pi}{c}K(\hat{x}\times\hat{z}) = -\frac{4\pi}{c}K\hat{y}$$

$$\Rightarrow \quad b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi K}{c}$$

define $\left.\begin{array}{l} b_y^> = b_{y0} + \delta b_y \\ b_y^< = b_{y0} - \delta b_y \end{array}\right\} \quad \delta b_y = -\frac{2\pi}{c}K$

$$\Rightarrow \quad \vec{B}^> = \vec{B}_0 - \frac{2\pi}{c}K\hat{y} \qquad \vec{B}_0 = b_{x0}\hat{x} + b_{y0}\hat{y} + b_{z0}\hat{z}$$
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi}{c}K\hat{y}$$

if $\vec{K}$ is the only source of magnetic field then $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi}{c}K\hat{y} & z>0 \\ \frac{2\pi}{c}K\hat{y} & z<0 \end{cases}$$