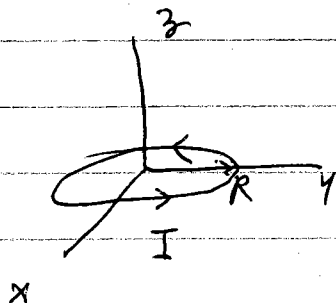


example circular current loop in xy plane
radius R



for $r > R$, $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} \phi_M$
where $\nabla^2 \phi_M = 0$.

Try Legendre polynomial expansion for ϕ_M

$$\phi_M = \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta) \quad (A_{\ell} \text{ terms vanish as want } B \rightarrow 0 \text{ as } r \rightarrow \infty)$$

$$\vec{B} = -\vec{\nabla} \phi_M = -\frac{\partial \phi_M}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi_M}{\partial \theta} \hat{\theta}$$

$$= \sum_{\ell} \left[\frac{(\ell+1)B_{\ell}}{r^{\ell+2}} P_{\ell}(\cos \theta) \hat{r} - \frac{B_{\ell}}{r^{\ell+2}} \frac{\partial P_{\ell}(\cos \theta)}{\partial \theta} \hat{\theta} \right]$$

$$\text{write } \frac{\partial P_{\ell}}{\partial \theta} = \frac{\partial P_{\ell}}{\partial x} \frac{\partial x}{\partial \theta} = -\frac{\partial P_{\ell}}{\partial x} \sin \theta \quad x = \cos \theta$$

$$= -P'_{\ell} \sin \theta$$

$$\vec{B} = \sum_{\ell} \left[\frac{(\ell+1)B_{\ell}}{r^{\ell+2}} P_{\ell}(\cos \theta) \hat{r} + \frac{B_{\ell}}{r^{\ell+2}} \sin \theta P'_{\ell}(\cos \theta) \hat{\theta} \right]$$

To determine the B_{ℓ} we compare with exact solution along $\hat{z}$ axis

$$\vec{B}(\hat{z}) = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{r^{\ell+2}} \hat{r} = \sum_{\ell} \frac{(\ell+1)B_{\ell}}{z^{\ell+2}} \hat{z}$$

since $P_{\ell}(1)=1$, $\sin(0)=0$ and $P'_{\ell}(1)$ finite, $\hat{r} = \hat{z}$ when $\theta=0$

exact solution on $\hat{z}$ axis:

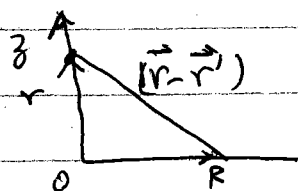
$$\vec{A} = \int \frac{d^3 r'}{c} \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A} = \int \frac{d^3 r'}{c} \vec{\nabla} \times \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\vec{B} = - \int \frac{d^3 r'}{c} \vec{j}(\vec{r}') \times \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$

$$\vec{B} = \int \frac{d^3 r'}{c} \vec{j}(\vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Biot-Savart Law for magnetostatics

For our loop



$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c} I \hat{\phi} \times \frac{[-R \hat{r} + z \hat{z}]}{(z^2 + R^2)^{3/2}}$$

↙ polar radial vector

$$\hat{r} \times \hat{\phi} = \hat{z}$$

$$= \int_0^{2\pi} d\phi \frac{R(I R)}{c} \frac{\hat{z}}{(z^2 + R^2)^{3/2}}$$

$\hat{\phi} \times \hat{z}$ then integrates to zero

$$\boxed{\vec{B}(z) = \frac{2\pi R^2 I}{c} \frac{\hat{z}}{(z^2 + R^2)^{3/2}}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I}{c} \frac{\hat{z}}{z^3} \frac{1}{\left(1 + \left(\frac{R}{z}\right)^2\right)^{3/2}} = \frac{2\pi R^2 I}{c} \frac{\hat{z}}{z^3} \left\{ 1 - \frac{3}{2} \left(\frac{R}{z}\right)^2 + \dots \right\}$$

$$= \frac{2\pi R^2 I}{c} \hat{z} \left\{ \frac{1}{z^3} - \frac{3}{2} \frac{R^2}{z^5} + \dots \right\}$$

$$= \left\{ \frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \dots \right\} \hat{z}$$

$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3 \pi R^2 I R^2}{4c}$$

So to order $L=3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 P_1(\cos\theta) \hat{r} + \sin\theta P_1'(\cos\theta) \hat{\theta}}{r^3} - \left[\frac{3 R^2 P_3(\cos\theta) \hat{r} + \frac{3}{4} R^2 \sin\theta P_3'(\cos\theta) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$$P_1(x) = x \Rightarrow P_1'(x) = 1$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \frac{1}{2}(15x^2 - 3)$$

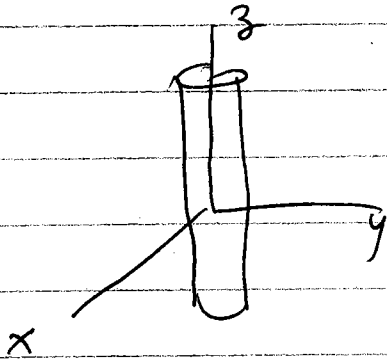
$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 \cos\theta \hat{r} + \sin\theta \hat{\theta}}{r^3} - \left[\frac{\frac{3}{2} R^2 (5 \cos^3\theta - 3 \cos\theta) \hat{r} + \frac{3}{8} R^2 \sin\theta (15 \cos^2\theta - 3) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$\frac{\pi R^2 I}{c} = m$ is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic ^{octapole} ~~quadrupole~~ term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5.40)

● example current carrying infinite cylinder radius R



- (i) $\vec{K} = K \hat{z}$ wire with surface current
 (ii) $\vec{K} = K \hat{\phi}$ solenoid

(c) $\vec{K} = K \hat{z}$

$2\pi R K = I$ total current
 "guess" + show it is correct

$r > R$ $\Phi_M = -\frac{4\pi R K}{c} \varphi$
 $r < R$ $\Phi_M = 0$

magnetic scalar potential $\nabla^2 \Phi_M = 0$

$r > R$ $\vec{B} = -\vec{\nabla} \Phi_M = -\frac{1}{r} \frac{\partial \Phi_M}{\partial \varphi} \hat{\phi} = \frac{4\pi R K}{c r} \hat{\phi} = \boxed{\frac{2I}{c r} \hat{\phi}}$ ← familiar result from Ampere
 $r < R$ $\vec{B} = 0$

$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \frac{2I}{c R} \hat{\phi} = \frac{4\pi K}{c} \frac{R}{R} \hat{\phi} = \frac{4\pi \vec{K}}{c} \times \hat{n}$
 where $\hat{n} = \hat{r}$
 as $\hat{z} \times \hat{r} = \hat{\phi}$

Note: $\Phi_M = -\frac{4\pi R K}{c} \varphi$ is not single valued!
 would not have found this using expansion of separation of coords in polar coords

(iii) $\vec{K} = K \hat{\phi}$

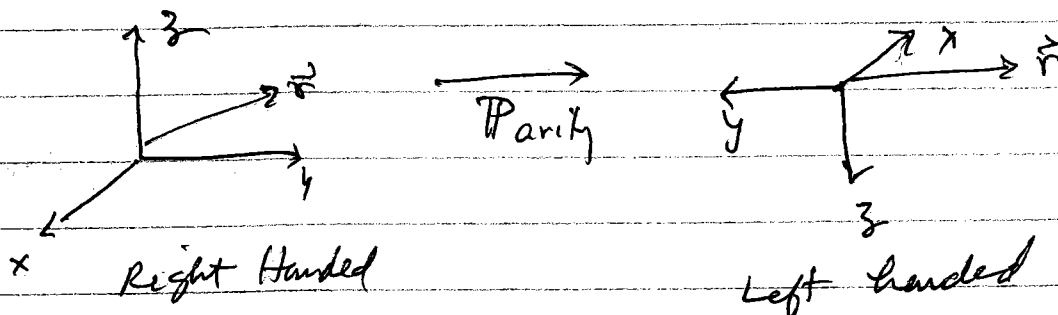
$r > R$ $\Phi_M = -B_1 z$
 $r < R$ $\Phi_M = -B_2 z$ } $\nabla^2 \Phi_M = 0$
 $r > R$ $\vec{B} = -\vec{\nabla} \Phi_M = B_1 \hat{z}$
 $r < R$ $\vec{B} = -\vec{\nabla} \Phi_M = B_2 \hat{z}$

$$\begin{aligned}
 \vec{B}_{\text{above}} - \vec{B}_{\text{below}} &= (B_1 - B_2) \hat{z} = \frac{4\pi}{c} \vec{K} \times \hat{n} \\
 &= \frac{4\pi}{c} K (\hat{\phi} \times \hat{r}) \\
 &= -\frac{4\pi}{c} K \hat{z}
 \end{aligned}$$

If current in solenoid is only source of $\vec{B}$ then expect $B_1 = 0$

$$\Rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c} K \hat{z}} \quad \text{familiar result}$$

Symmetry under parity transformation vector vs. pseudo vector



$$\vec{r} = (x, y, z) \rightarrow (-x, y, -z)$$

$$P(\vec{r}) = -\vec{r} \quad \text{position } \vec{r} \text{ is odd under parity}$$

Any vector-like quantity that is odd under P is a vector.

examples of vectors

position $\vec{r}$

velocity $\vec{v} = \frac{d\vec{r}}{dt}$

acceleration $\vec{a} = \frac{d\vec{v}}{dt}$

since $\vec{r}$ is vector and t is scalar
 $P(t) = t$

Force $\vec{F} = m\vec{a}$

since $\vec{a}$ is vector and m is scalar

momentum $\vec{p} = m\vec{v}$

since $\vec{v}$ is vector and m is scalar

electric field $\vec{E} = \vec{F} / q$

since $\vec{F}$ is vector and q is scalar

$$P(q) = q$$

current $\vec{j} = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i(t))$

any vector-like quantity that is even under P is a pseudovector

angular momentum $\vec{L} = \vec{r} \times \vec{p}$ since $\vec{r} \rightarrow -\vec{r}$ and $\vec{p} \rightarrow \vec{p}$,
 $\vec{L} \rightarrow \vec{L}$ under P

$\vec{L}$ is even under P

magnetic field $\vec{B} = \frac{1}{q} \vec{v} \times \vec{p}$ since $\vec{v}$ and $\vec{p}$ are vectors and q is scalar, $\vec{B}$ must be pseudovector.

cross product of any two vectors is a pseudovector
 " " " vector and pseudovector is a vector

when solving for $\vec{E}$, it can only be made up of vectors that exist in the problem

when solving for $\vec{B}$, it can only be made up of pseudovectors that exist in the problem

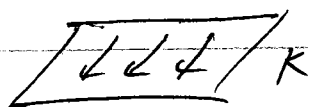
ex charged plane



only directions in problem is normal $\hat{n}$
 $\hat{n}$ is a vector

$$\vec{E} \propto \hat{n}$$

surface current



only directions are the vectors $\hat{n}$ and $\vec{K}$. But $\vec{B}$ can only be made of pseudovectors

$$\Rightarrow \vec{B} \propto (\vec{K} \times \hat{n})$$

Dielectrics + Magnetic Materials - Macroscopic Maxwell Equ

Dielectrics

Maxwell's equations apply exactly to the free microscopic electric and magnetic fields that arise from all charges and currents.

$$\vec{\nabla} \cdot \vec{b} = 0 \quad \vec{\nabla} \times \vec{e} + \frac{1}{c} \frac{\partial \vec{b}}{\partial t} = 0$$

$$\vec{\nabla} \cdot \vec{e} = 4\pi \rho_0 \quad \vec{\nabla} \times \vec{b} = \frac{4\pi}{c} \vec{j}_0 + \frac{1}{c} \frac{\partial \vec{e}}{\partial t}$$

where $\vec{e}$ and $\vec{b}$ are microscopic fields from total charge density ρ_0 and current density $\vec{j}_0$.

However, in most problems involving macroscopic objects, if we took ρ_0 and $\vec{j}_0$ to describe charge + current of each individual atom in a material, then they, and the resulting $\vec{e}$ and $\vec{b}$ would be enormously complicated functions varying rapidly over distances $\sim 10^{-8}$ cm and times $\sim 10^{-16}$ sec.

In classical EM we are generally concerned with phenomena that vary extremely slowly compared to these length + time scales,

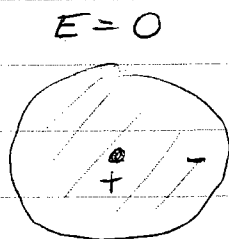
Rather than worry about the microscopic details of ρ and $\vec{j}$ as resulting $\vec{E}$ and $\vec{B}$ we want to describe phenomena in terms of averaged ~~smoothly varying~~ ~~from~~ averaged quantities that are smoothly varying at the atomic scale. This results in what are known as the macroscopic Maxwell equations.

Dielectric Materials

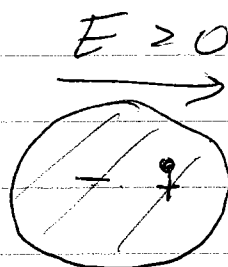
can be solid, liquid or gas

A dielectric material is an insulator. valence Electrons are bound to the ionic cores of the atoms.

When no electric field is present, the averaged ρ in the dielectric vanishes! One might therefore think that electrodynamics in a dielectric is just due to whatever "extra" or "free" charge is added to the dielectric. However this is not true due to the phenomena of "polarization".



electron cloud
centered on ionic
~~nucleus~~ core
dipole moment
vanishes



electron cloud
and ionic core
displaced $\vec{r} \propto \vec{E}$
atom is "polarized"
has dipole moment $\vec{p} = q\vec{r} \propto q\vec{E}$

$$\vec{p} = \alpha \vec{E}$$

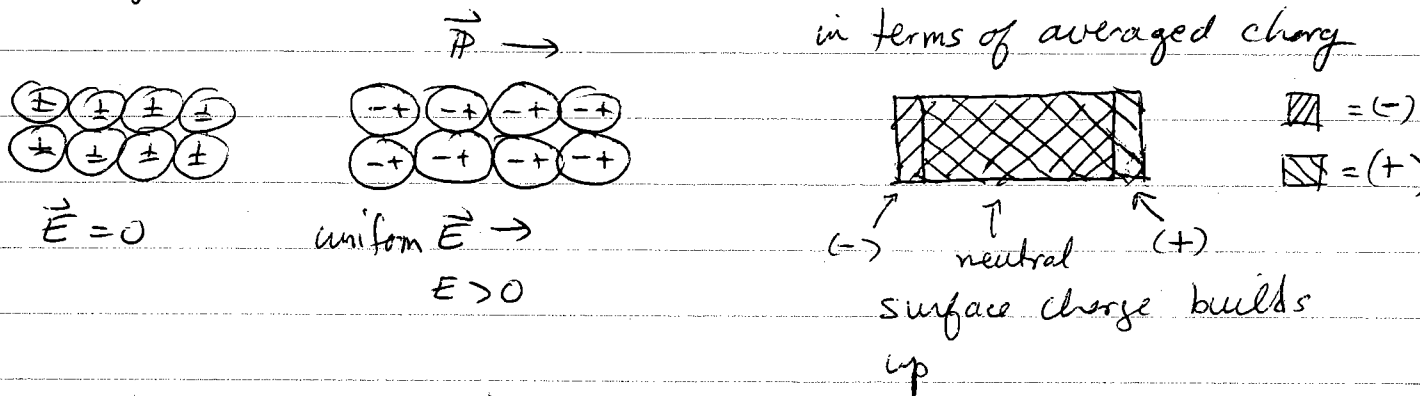
↑
atomic
polarizability

Polarization density $\vec{P}(\vec{r}) = \sum_i \vec{p}_i \delta(\vec{r} - \vec{r}_i)$

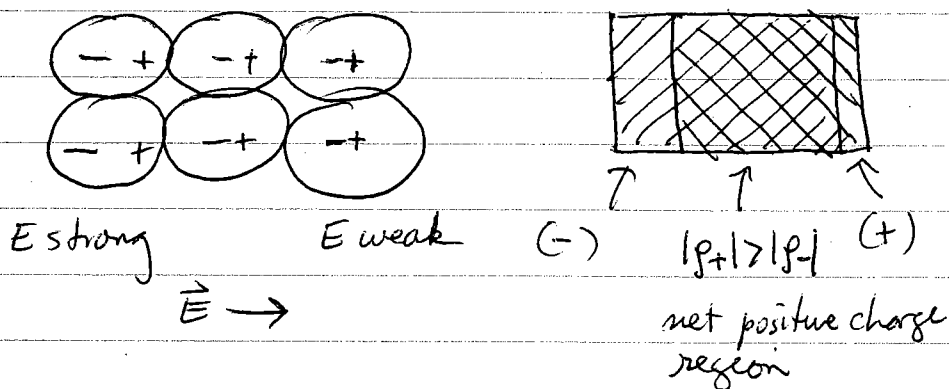
$\vec{p}_i$
dipole moment of atom i
at position $\vec{r}_i$

Polarization density $\vec{P}$ can give rise to regions of net charge - sometimes called "bound charge"

Example



For a non uniform $\vec{E}$, atoms are more strongly polarized where E is largest



For uniform $\vec{P}$, build up surface charge σ_b
For nonuniform $\vec{P}$, also can build up vol charge density ρ_b .