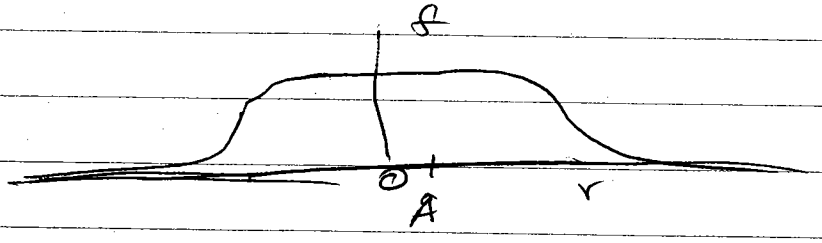
We now carry out the averaging explicitly to see how such polarization enters the macroscopic Maxwell equations

(Jackson 6.6)

Define spatially averaged quantities by

$$\langle F(\vec{r},t)\rangle = \int d^3r'\, f(\vec{r}')\, F(\vec{r}-\vec{r}',t)$$

where $f(\vec{r})$ vanishes for $|\vec{r}|$ large on microscopic length scales, but short on macroscopic length scales. $f(\vec{r})$ normalized to unity $\int d^3r\, f(\vec{r}) = 1$. Other details of $f(\vec{r})$ are not too important, as long as $f(\vec{r})$ is a smooth function of $\vec{r}$



want $f \approx 1$ for $r < R$

$f \approx 0$ for $r \gg R$

where R is length scale in between micro + macro

$$\frac{\partial}{\partial r_i}\langle F(\vec{r},t)\rangle = \int d^3r'\, f(\vec{r}')\, \frac{\partial F(r-r')}{\partial r_i} = \left\langle \frac{\partial f}{\partial r_i} \right\rangle$$

$$\frac{\partial}{\partial t}\langle F(\vec{r},t)\rangle = \left\langle \frac{\partial F}{\partial t} \right\rangle$$

Define the macroscopic fields

$$\vec{E}(\vec{r},t) \equiv \langle \vec{e}(\vec{r},t)\rangle$$
$$\vec{B}(\vec{r},t) \equiv \langle \vec{b}(\vec{r},t)\rangle$$

Then $\vec{\nabla}\cdot\vec{b} = 0 \Rightarrow \langle \vec{\nabla}\cdot\vec{b} \rangle = 0$

$\Rightarrow \vec{\nabla}\cdot\langle\vec{b}\rangle = 0$

$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

$$\vec{\nabla}\times\vec{e} + \frac{\partial\vec{b}}{\partial t} = 0 \Rightarrow \vec{\nabla}\times\langle\vec{e}\rangle + \frac{\partial}{\partial t}\langle\vec{b}\rangle = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

Remaining Maxwell eqn, upon averaging, become

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle\rho_0\rangle$$
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle\vec{j}_0\rangle + \frac{\partial\vec{E}}{\partial t}$$

Consider $\langle\rho_0\rangle$

$\rho_0 = \sum_i q_i\,\delta(\vec{r} - \vec{r}_i(t))$ sum over all charges

Consider dividing the charge into "free" charges and "bound" charges, where the latter are associated with the molecules that make up the dielectric

$\rho_{free} = \sum_{i\ free} q_i\,\delta(\vec{r} - \vec{r}_i(t))$ sum over only free charges

$\rho_{bound} = \sum_n \rho_n(\vec{r}, t)$

↑ charge distribution of molecule n

$\rho_n = \sum_{i\in n} q_i\,\delta(\vec{r} - \vec{r}_i(t))$ sum over charges in molecule n

$$\langle \rho_n(\vec{r},t)\rangle = \int d^3r'\, f(\vec{r}')\, \rho_n(\vec{r}-\vec{r}',t)$$

$$= \sum_{i\in n} q_i \int d^3r'\, f(\vec{r}')\, \delta(\vec{r}-\vec{r}' - \vec{r}_i(t))$$

$$= \sum_{i\in n} q_i\, f(\vec{r}-\vec{r}_i(t))$$

write $\vec{r}_i(t) = \vec{r}_n(t) + \vec{r}_{ni}(t)$

↑ position of center of ~~mass~~ mass of molecule n

↑ position of charge i of molecule n with respect to center of ~~mass~~ mass

$$\langle \rho_n(\vec{r},t)\rangle = \sum_{i\in n} q_i\, f(\vec{r}-\vec{r}_n-\vec{r}_{ni})$$

Since the $|\vec{r}_{ni}|$ are all of atomic length scale, and f is slowly varying on this length scale, we can expand

$$\langle \rho_n(\vec{r},t)\rangle = \sum_{i\in n} q_i \Big[f(\vec{r}-\vec{r}_n) - \big(\vec{\nabla} f(\vec{r}-\vec{r}_n)\big)\cdot \vec{r}_{ni} + \frac{1}{2}\sum_{\alpha\beta=1}^{3} \frac{\partial f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta} (\vec{r}_{ni})_\alpha (\vec{r}_{ni})_\beta + \cdots \Big]$$

$$= f(\vec{r}-\vec{r}_n)\Big[\sum_{i\in n} q_i\Big]$$

$$- \big(\vec{\nabla} f(\vec{r}-\vec{r}_n)\big)\cdot \sum_{i\in n} q_i \vec{r}_{ni}$$

$$+ \sum_{\alpha,\beta=1}^{3} \left(\frac{1}{6}\frac{\partial f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta}\right) 3\sum_{i\in n} q_i (r_{ni})_\alpha (r_{ni})_\beta$$

Define $q_n \equiv \sum_{i \in n} q_i$ total charge molecule n

$\vec{p}_n \equiv \sum_{i \in n} q_i \vec{r}_{ni}$ dipole moment about center of mass of molec n

$\overleftrightarrow{Q}'_n \equiv \sum_{i \in n} 3 q_i \vec{r}_{ni} \vec{r}_{ni}$ quadrapole moment about center of mass of molec n

(prime on $\overleftrightarrow{Q}'$ since definition here is a little different from that of multipole exp)

$$\langle \rho_n(\vec{r},t) \rangle = q_n f(\vec{r}-\vec{r}_n) - \vec{p}_n \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n) + \frac{1}{6} \sum_{\alpha\beta} (Q'_n)_{\alpha\beta} \frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta}$$

Now use $\langle \delta(\vec{r}-\vec{r}_n) \rangle = f(\vec{r}-\vec{r}_n)$ by definition of averaging

$$\Rightarrow \langle \rho_n(\vec{r},t) \rangle = \langle q_n \delta(\vec{r}-\vec{r}_n) \rangle - \vec{\nabla} \cdot \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle + \frac{1}{6} \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \langle (Q'_n)_{\alpha\beta} \delta(\vec{r}-\vec{r}_n) \rangle$$

$\vec{\nabla} \cdot p\delta = \vec{p} \cdot \vec{\nabla}\delta$

Now

$$\langle \rho_{bound}(\vec{r},t) \rangle = \sum_n \langle \rho_n(\vec{r},t) \rangle = \left\langle \sum_n q_n \delta(\vec{r}-\vec{r}_n) \right\rangle - \vec{\nabla} \cdot \left\langle \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n) \right\rangle + \frac{1}{6} \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \left\langle \sum_n (Q_n)'_{\alpha\beta} \delta(\vec{r}-\vec{r}_n) \right\rangle$$

Define $\vec{P}(\vec{r},t) \equiv \left\langle \sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n) \right\rangle$ average polarization density

$\overleftrightarrow{Q}'(\vec{r},t) \equiv \frac{1}{6}\left\langle \sum_n \overleftrightarrow{Q}'_n \delta(\vec{r}-\vec{r}_n) \right\rangle$ average quadrapole density

$$\langle \rho_{bound} \rangle = \sum_n \langle q_n \delta(\vec{r}-\vec{r}_n) \rangle - \vec{\nabla}\cdot\vec{P} + \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \overleftrightarrow{Q}'_{\alpha\beta}$$

~~Denote~~

Define the macroscopic charge density

$$\rho \equiv \left\langle \sum_{i\ free} q_i \delta(\vec{r}-\vec{r}_i) \right\rangle + \left\langle \sum_{n\ molec} q_n \delta(\vec{r}-\vec{r}_n) \right\rangle$$

Then

$$\vec{\nabla}\cdot\vec{E} = 4\pi\langle \rho_0 \rangle = 4\pi\left[\rho - \vec{\nabla}\cdot\vec{P} + \sum_{\alpha\beta} \frac{\partial^2}{\partial r_\alpha \partial r_\beta} \overleftrightarrow{Q}'_{\alpha\beta}\right]$$

$$\sum_\alpha \frac{\partial}{\partial r_\alpha}\left[E_\alpha + 4\pi P_\alpha - 4\pi \sum_\beta \frac{\partial}{\partial r_\beta} Q'_{\alpha\beta} \right] = 4\pi\rho$$

Define <u>electric displacement vector</u>

$$D_\alpha = E_\alpha + 4\pi P_\alpha - 4\pi \sum_\beta \frac{\partial}{\partial r_\beta} Q'_{\alpha\beta}$$

then $\boxed{\vec{\nabla}\cdot\vec{D} = 4\pi\rho}$

In most materials, the quadrapole and higher terms are negligible and we can take

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

Why quadrupole + higher order terms can generally be ignored:

Let a_0 be the length scale that characterizes the size of a molecule in the dielectric
Let ℓ be the typical spacing between molecules
Let L be the length scale of the spatial averaging function $f(\vec{r})$. $L \gg a_0$

dipole moment per a_0
$\Rightarrow$ polarization $P \sim \frac{a_0}{\ell^3}$

$\vec{\nabla} \cdot \vec{P} \sim \left(\frac{a_0}{L}\right) \frac{1}{\ell^3}$ since P cannot vary on length scale shorter than the averaging length L

quadrupole moment

$$Q \sim a_0^2$$

quadrupole density $\mathcal{Q} \sim \frac{a_0^2}{\ell^3}$

$$\frac{\partial^2 \mathcal{Q}}{\partial r_\alpha \partial r_\beta} \sim \left(\frac{a_0}{L}\right)^2 \frac{1}{\ell^3}$$

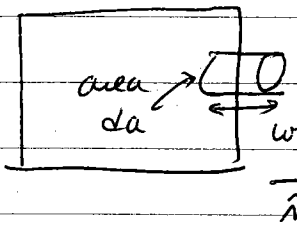
each higher moment gives extra factor a_0
each higher derivative gives extra factor $\frac{1}{L}$

So quadrupole is smaller than dipole term by factor $\left(\frac{a_0}{L}\right) \ll 1$. Higher terms smaller by additional factors of $\left(\frac{a_0}{L}\right)$

For insulators one generally has $\rho_{free} = 0$
molecules are neutral.
In this case the macroscopic ρ is just the free charge $\rho = \langle \rho_{free} \rangle$.
And the bound charge is just

$$\boxed{\langle \rho_{bound} \rangle = -\vec{\nabla} \cdot \vec{P}}$$

At a surface of a dielectric



$$-\int_V d^3r\ \vec{\nabla} \cdot \vec{P} = -\oint_S da\ \hat{n} \cdot \vec{P}$$

contrib from sides $\to 0$ as $\ell \to 0$
contrib from outside surface $= 0$ as $P = 0$ outside
only contrib is from inside surface ($\hat{n}$ is outward normal

$$= \hat{n} \cdot \vec{P}\, da$$

$$= \int_V d^3r\ \rho_{bound}$$

as $\ell \to 0$, $\int_V d^3r\ \rho_{bound} \to \int da\ \sigma_{bound} = da\, \sigma_{bound}$ surface charge

$$\Rightarrow \boxed{\sigma_{bound} = \hat{n} \cdot \vec{P}}$$ at surface of dielectric

The

Magnetic Materials

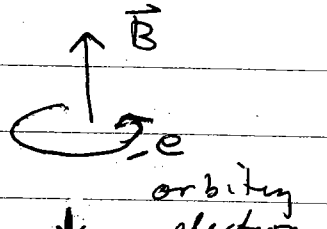
Circulating currents on atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material.

Sources of circulating atomic currents:

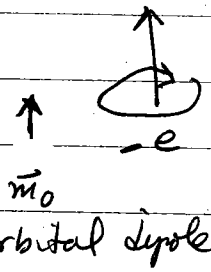
1) intrinsic angular momentum of electrons, ie "electron spin" - can add up and give a net angular momentum to atom
2) orbital angular momentum of electrons - can add up to give net angular momentum of atom.

(1) + (2) $\Rightarrow$ atoms can have a net magnetic dipole moment. When $\vec{B}=0$, these atomic moments are generally in random orientations and average to zero (exception is a ferromagnet where moments can allign even if $\vec{B}=0$) When apply $\vec{B} \neq 0$, the moments tend to allign parallel to $\vec{B}$ giving a net magnetization density $\vec{M} \propto \vec{B}$. This is a paramagnetic effect.

But there is also a diamagnetic effect from orbital angular momentum (exists even if total angular momentum of electrons is zero, ie exists for atoms with zero net dipole moment)



← applying $\vec{B}$ to orbiting electron speeds up its orbital velocity. Increased angular momentum of negatively charged electron gives change in dipole moment $\Delta\vec{m} \propto -\vec{B}$



← applying $\vec{B}$ to orbiting electron slows down its orbital velocity. Net result is again that $\Delta\vec{m} \propto -\vec{B}$

see Griffiths chpt 6 + prob 7.17 2nd ed for details

No matter which way electron orbits with respect to $\vec{B}$, result is a decrease in magnetic moment, so $\Delta\vec{m} \propto -\vec{B}$

That $\Delta\vec{m}$ is opposite to $\vec{B}$ is called diamagnetism

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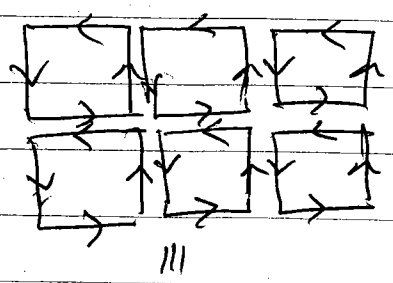
Model atomic magnetic moments as small current loops. When loops get oriented, ie there is non zero average magnetization density

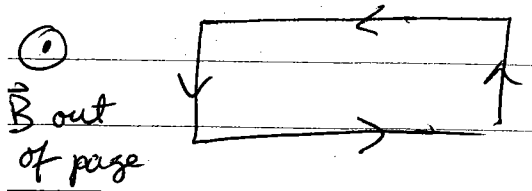
$$\vec{M}(\vec{r}) = \sum_i \vec{m}_i \, \delta(\vec{r} - \vec{r}_i)$$

then net effect is to have a current flowing around the system. This current gives rise to magnetic fields

allegied atomic moments in a uniform applied $\vec{B}$

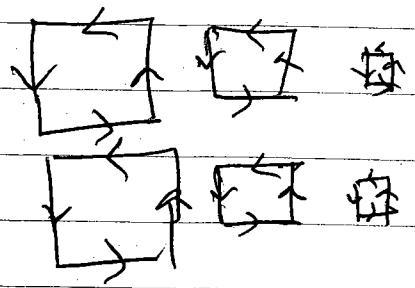


in interior, currents in opposite directions cancell also $\vec{j} = 0$ inside



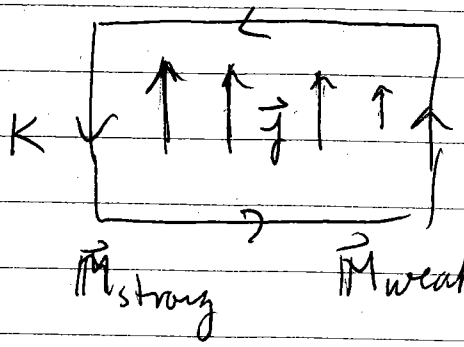
but is net circulation of current around bounday of material
$\Rightarrow$ surface current $\vec{K}_{bound}$

If $\vec{B}$ is not uniform, then $\vec{M}$ is not uniform
Can create finite current density $\vec{j}$ in interior, as well as surface currents



Now currents in interior do not cancell. Net current $\vec{j}_{bound}$ in interior

$\vec{B}$ strong $\vec{B}$ weak



$\vec{B}$ out of page $\Rightarrow$ $\vec{M}$ out of page
~~$\vec{M}$ varies along page~~
$\vec{M}$ varies in direction $\perp$ direction of $\vec{M}$
$\Rightarrow \vec{\nabla} \times \vec{M} \neq 0$ gives $\vec{j}_{bound}$