Average current

$$\langle \vec{j}_0 \rangle = \langle \sum_{i \in \text{free}} q_i \vec{v}_i \, \delta(\vec{r}-\vec{r}_i) \rangle + \sum_{n \text{ molec}} \langle \vec{j}_n \rangle$$

current from free charges

current from molecule n of the dielectric

$$\langle \vec{j}_n(\vec{r},t) \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \langle \delta(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t)) \rangle$$

$$= \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \, f(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t))$$

$\vec{v}_n = \frac{d\vec{r}_n}{dt}$ $\quad$ $\vec{v}_{ni} = \frac{d\vec{r}_{ni}}{dt}$ $\quad$ position of CM of molec n $\quad$ position of charge i wrt CM

as with $\langle \rho_0 \rangle$, we can expand in $\vec{r}_{ni}$

$$\langle \vec{j}_n \rangle = \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \Big\{ f(\vec{r}-\vec{r}_n) - \vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n) + \frac{1}{2} \sum_{\alpha\beta} (r_{ni})_\alpha (r_{ni})_\beta \frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta} + \cdots \Big\}$$

we will keep only the first two terms in the expansion

The various terms we have to consider are

① $\sum_{i \in n} q_i \vec{v}_n \, \delta(\vec{r} - \vec{r}_n)$

② $\sum_{i \in n} q_i \vec{v}_{ni} \, \delta(\vec{r} - \vec{r}_n)$

③ $-\sum_{i \in n} q_i \vec{v}_n \left[\vec{r}_{ni} \cdot \vec{\nabla} \delta(\vec{r} - \vec{r}_n) \right]$

④ $-\sum_{i \in n} q_i \vec{v}_{ni} \left[\vec{r}_{ni} \cdot \vec{\nabla} \delta(\vec{r} - \vec{r}_n) \right]$

$$① = \vec{v}_n \, \delta(\vec{r} - \vec{r}_n) \sum_{i \in n} q_i = q_n \vec{v}_n \, \delta(r - r_n)$$
$$= \langle q_n \vec{v}_n \, \delta(\vec{r} - \vec{r}_n) \rangle$$

this is just current of molecule as if it were a point charge q_n. For a neutral molecule $q_n = 0$ and this term vanishes.

② Note: $$\frac{\partial}{\partial t} \langle \vec{p}_n \, \delta(\vec{r} - \vec{r}_n) \rangle = \frac{\partial}{\partial t} \left(\sum_{i \in n} q_i \vec{r}_{ni} \, \delta(\vec{r} - \vec{r}_n) \right)$$
$$= \sum_{i \in n} q_i \vec{v}_{ni} \, \delta(\vec{r} - \vec{r}_n)$$
$$+ \sum_{i \in n} q_i \vec{r}_{ni} \left[-\vec{\nabla} \delta(\vec{r} - \vec{r}_n) \cdot \vec{v}_n \right]$$

So for ②, $$\sum_{i \in n} q_i \vec{v}_{ni} \, \delta(\vec{r} - \vec{r}_n)$$
$$= \frac{\partial}{\partial t} \langle \vec{p}_n \, \delta(\vec{r} - \vec{r}_n) \rangle$$
$$+ \left[\vec{v}_n \cdot \vec{\nabla} \delta(\vec{r} - \vec{r}_n) \right] \vec{p}_n$$

So

$$② = \sum_{i\in n} q_i \vec{v}_{ni} f(\vec{r}-\vec{r}_n) = \frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle + (\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(r-r_n)\rangle$$

2nd term is $\sum_\alpha v_{n\alpha} \frac{\partial}{\partial r_\alpha}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$

$$③ = -\vec{v}_n\Big(\sum_{i\in n} q_i \vec{r}_{ni}\Big)\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n) = -\vec{v}_n\left(\vec{p}_n\cdot\vec{\nabla} f(\vec{r}-\vec{r}_n)\right)$$

$$= -\vec{v}_n\,\vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle = \sum_\alpha \vec{v}_n \frac{\partial}{\partial r_\alpha}\langle p_{n\alpha}\delta(\vec{r}-\vec{r}_n)\rangle$$

$$④ = -\vec{\nabla} f(\vec{r}-\vec{r}_n)\cdot \sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni}$$

We have seen the tensor $\sum_i q_i \vec{r}_{ni}\vec{v}_{ni}$ before when we considered the magnetic dipole moment

$$\sum_{i\in n} q_i \vec{r}_{ni}\vec{v}_{ni} = \int d^3r\, \vec{r}\,\vec{j}$$ where $\vec{j}(\vec{r}) \equiv \sum_{i\in n} q_i \vec{v}_{ni}\delta(\vec{r}-\vec{r}_{ni})$ is current density with respect to center of mass of molecule

We had $\int d^3r\, \vec{r}\,\vec{j} = -\int d^3r\, \vec{j}\,\vec{r} - \int d^3r\,(\vec{\nabla}\cdot\vec{j})\,\vec{r}\,\vec{r}$

in statics, $\vec{\nabla}\cdot\vec{j}=0$

in general $\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$

$$\int d^3r\, \vec{r}\,\vec{j} = -\int d^3r\, \vec{j}\,\vec{r} + \int d^3r\, \frac{\partial\rho}{\partial t}\vec{r}\,\vec{r}$$

$$= -\int d^3r\, \vec{j}\,\vec{r} + \frac{\partial}{\partial t}\left[\int d^3r\, \rho\, \vec{r}\,\vec{r}\right]$$

although this is not zero, it is a <u>quadrapole</u> term of the same order as the terms we dropped when we truncated expansion to <u>linear order</u>

$$\sim O\left(\frac{a_0}{L}\right)^2$$

So $\int d^3r\, \vec{r}\,\vec{J} \simeq -\int d^3r\, \vec{J}\,\vec{r}$ ignoring the quadrapole term

$$= \frac{1}{2}\int d^3r \left[\vec{r}\,\vec{J} - \vec{J}\,\vec{r}\right]$$

$$\sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = \frac{1}{2}\sum_{i\in n} q_i \left[\vec{r}_{ni}\,\vec{v}_{ni} - \vec{v}_{ni}\,\vec{r}_{ni}\right]$$

$$-\vec{\nabla} f(r-r_n)\cdot \sum_{i\in n} q_i \vec{r}_{ni}\,\vec{v}_{ni} = -\vec{\nabla} f(\vec{r}-\vec{r}_n)\cdot \frac{1}{2}\sum_i \left[\vec{r}_{ni}\vec{v}_{ni} - \vec{v}_{ni}\vec{r}_{ni}\right]$$

$$= \frac{-1}{2}\sum_{i\in n} q_i\left[(\vec{\nabla} f\cdot \vec{r}_{ni})\vec{v}_{ni} - (\vec{\nabla} f\cdot \vec{v}_{ni})\vec{r}_{ni}\right]$$

$$= -\frac{1}{2}\sum_{i\in n} q_i\, \vec{\nabla} f \times (\vec{v}_{ni}\times \vec{r}_{ni})$$ triple product rule

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n)\times \frac{1}{2}\sum_{i\in n} \vec{r}_{ni}\times \vec{v}_{ni}\, q_i$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n)\times \frac{1}{2}\int d^3r\, \vec{r}\times\vec{J}$$

$$= \vec{\nabla} f(\vec{r}-\vec{r}_n)\times c\,\vec{m}_n$$ where $\vec{m}_n = \frac{1}{2c}\sum_{i\in n} \vec{r}_{ni}\times \vec{v}_{ni}\, q_i$

is magnetic dipole moment of molecule n

$$= \vec{\nabla}\times f(\vec{r}-\vec{r}_n)\, c\,\vec{m}_n$$

$$= \vec{\nabla}\times \left\langle c\,\vec{m}_n\, \delta(\vec{r}-\vec{r}_n)\right\rangle$$

Adding all the pieces

$$\langle \vec{j}_n \rangle = \underset{①}{\langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n)\rangle} + c\vec{\nabla}\times \underset{④}{\langle \vec{m}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

$$+ \underset{②}{\frac{\partial}{\partial t}\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle} + \underset{②}{(\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

$$- \underset{③}{\vec{v}_n \vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle}$$

Define $\vec{M}(\vec{r}) \equiv \sum_n \langle \vec{m}_n \delta(\vec{r}-\vec{r}_n)\rangle$ average magnetization density

$\vec{P}(\vec{r}) \equiv \sum_n \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle$ polarization density, as before

$$\sum_n \langle \vec{j}_n \rangle = \sum_n \langle q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n)\rangle + c\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

$$+ \sum_n \left[(\vec{v}_n\cdot\vec{\nabla})\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle - \vec{v}_n \vec{\nabla}\cdot\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n)\rangle\right]$$

see Jackson (6.96) for additional electric quadropole terms

The last term on the right hand side is usually small and ignored. This is because the molecular velocities $\vec{v}_n$ are usually small, and randomly oriented, so that they average to zero.
(see Jackson (6.100) for case of net translation of dielectric, $\vec{v}_n$ = const all n)

Define macroscopic current density

$$\vec{j}(\vec{r},t) = \Big\langle \sum_{i \in free} q_i \vec{v}_i \, \delta(\vec{r}-\vec{r}_i) \Big\rangle + \Big\langle \sum_{\substack{n \\ molecs}} q_n \vec{v}_n \, \delta(\vec{r}-\vec{r}_n) \Big\rangle$$

current of free charges — current of molecular drifting if molecules are charged

Then $\langle \vec{j}_0 \rangle = \vec{j} + c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$

Ampere's law becomes upon averaging

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\langle \vec{j}_0 \rangle + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$= \frac{4\pi}{c}\vec{j} + 4\pi\vec{\nabla}\times\vec{M} + \frac{4\pi}{c}\frac{\partial \vec{P}}{\partial t} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\left(\vec{B} - 4\pi\vec{M}\right) = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E} + 4\pi\vec{P}\right)$$

define $\boxed{\vec{H} \equiv B - 4\pi\vec{M}}$ to get

$$\boxed{\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}}$$

$\boxed{\vec{D} = \vec{E} + 4\pi\vec{P}}$ as before

official nomenclature: $\vec{B}$ is the magnetic induction

$\vec{H}$ is the magnetic field

common usage: both $\vec{H}$ and $\vec{B}$ are called magnetic field

When atoms have intrinsic magnetic moments due to electron spin, we can add these to $\vec{M}$ in obvious way

When molecules are neutral, $q_n = 0$, the "bound current" is given by

$$\vec{J}_{bound} = \sum_n \langle \vec{j}_n \rangle = c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of bound charge

$$\vec{\nabla}\cdot\vec{J}_{bound} = c\,\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) + \vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t}$$

$$= 0 + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P})$$

$$= -\frac{\partial \rho_{bound}}{\partial t}$$ where $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$ is bound charge density

So $$\boxed{\vec{\nabla}\cdot\vec{J}_{bound} + \frac{\partial \rho_{bound}}{\partial t} = 0}$$

and bound charge is conserved.

Since total average charge must be conserved, ie

$$\vec{\nabla}\cdot\langle\vec{j}_0\rangle - \frac{\partial\langle\rho_0\rangle}{\partial t} = 0,$$ and $\langle\vec{j}_0\rangle = \vec{J} + \vec{J}_{bound}$

$\vec{J}$ ← free current

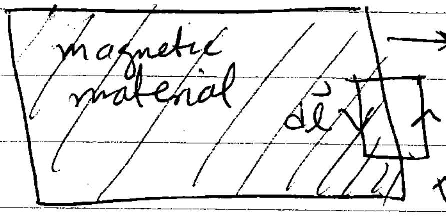
$\langle\rho_0\rangle = \rho + \rho_{bound}$

ρ ← free charge

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

free charge is also conserved

At a surface of a magnetic material



$\to \hat{m}$ outward normal to surface

take $\hat{z} \equiv d\hat{l} \times \hat{m}$ out of page

Amperean loop C bounding surface S of area da

$$c\int_S da\, \hat{z}\cdot(\vec{\nabla}\times\vec{M}) = \int_S da\, \hat{z}\cdot\vec{J}_{bound} = da\, \hat{z}\cdot\vec{J}_{bound}$$

$$= (d\vec{l}\times\hat{m})\cdot\vec{K}_{bound} \quad \text{as width of loop } \to 0$$

$$= (\hat{m}\times\vec{K}_{bound})\cdot d\vec{l}$$

But by Stokes theorem

$$c\int_S da\, \hat{z}\cdot(\vec{\nabla}\times\vec{M}) = c\int_C d\vec{l}\cdot\vec{M} = c\, d\vec{l}\cdot\vec{M} \quad \text{since width} \to 0 \text{ and } \vec{M}=0 \text{ outside}$$

$$\Rightarrow \quad c\, d\vec{l}\cdot\vec{M} = (\hat{m}\times\vec{K}_{bound})\cdot d\vec{l} \quad \text{for any } d\vec{l} \text{ in plane of surface}$$

$$\Rightarrow \quad c\vec{M}_t = \hat{m}\times\vec{K}_{bound}$$

where $\vec{M}_t$ is component of $\vec{M}$ tangential to the surface (since $\vec{K}_b$ is in plane of surface, $\hat{m}\times\vec{K}$ is also entirely in the plane of the surface)

$$\Rightarrow \quad c\,\hat{m}\times\vec{M}_t = c\,\hat{m}\times\vec{M} = \hat{m}\times(\hat{m}\times\vec{K}_{bound}) = -\vec{K}_{bound}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \vec{K}_{bound} &= c\,\vec{M}\times\hat{m} \\ \vec{J}_{bound} &= c\,\vec{\nabla}\times\vec{M} \end{aligned}}$$

Total bound charge vanishes

$$Q_{bound} = \int_V d^3r\, \rho_{bound} + \int_S da\, \sigma_{bound}$$

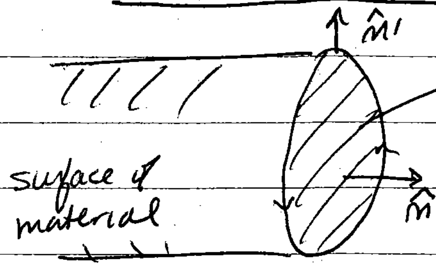
↑ vol of dielectric ← surface of dielectric

$$= \int_V d^3r\, -\vec{\nabla}\cdot\vec{P} + \int da\, \hat{n}\cdot\vec{P}$$

but by Gauss theorem $\int_V d^3r\, \vec{\nabla}\cdot\vec{P} = \int da\, \hat{n}\cdot\vec{P}$

so $Q_{bound} = -\int da\, \hat{n}\cdot\vec{P} + \int da\, \hat{n}\cdot\vec{P} = 0$

Total bound current vanishes



crossectional area of magnetic material

S - bounding curve is C

$\hat{n}$ is normal to crossection

$\hat{n}'$ is normal to surface

total current flowing through S is

$$\int_S da\, \hat{n}\cdot\vec{j}_{bound} + \int_C dl\, \vec{K}_{bound}\cdot\hat{n}$$

$$= c\int_S da\, \hat{n}\cdot(\vec{\nabla}\times\vec{M}) + c\int_C dl\, \hat{n}\cdot(\vec{M}\times\hat{n}')$$

$$= c\int_C d\vec{l}\cdot\vec{M} + c\int_C dl\, \underbrace{(\hat{n}'\times\hat{n})}_{=-\hat{t}}\cdot\vec{M}$$

$= -\hat{t}$ unit tangent, $d\vec{l} = dl\,\hat{t}$

$$= c\int_C d\vec{l}\cdot\vec{M} - c\int_C d\vec{l}\cdot\vec{M} = 0$$