Conservation of Energy

– leave macroscopic Maxwell eqns for present. $\vec{E}, \vec{B}, \rho, \vec{j}$ are now the exact microscopic quantities

Consider a collection of charged particles, described by charge density ρ and current density $\vec{j}$. The particles are contained in a volume V.

Define $\mathcal{E}_{mech}$ as total "mechanical" energy of the particles. $\mathcal{E}_{mech}$ = sum of particles kinetic energy plus potential energy of any non electromagnetic forces.

The particles will exert forces on each other via their electromagnetic interactions, ie via the $\vec{E}$ and $\vec{B}$ fields that they create. Define W as the work done on the particles by all electromagnetic forces. Then, by the work energy theorem of mechanics:

$$\frac{d\mathcal{E}_{mech}}{dt} = \frac{dW}{dt}$$

For a single charge q_i, (at $\vec{r}_i$ with velocity $\vec{v}_i$) $\quad \frac{dW}{dt} = \vec{F}_i \cdot \vec{v}_i$

$$= q_i \vec{E}(\vec{r}_i) \cdot \vec{v}_i + \underbrace{q\left(\frac{\vec{v}_i}{c} \times \vec{B}\right) \cdot \vec{v}_i}_{=0}$$

$$= q_i \vec{E}(r_i) \cdot \vec{v}_i$$

For the collection of charges, with

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i\, \delta(\vec{r}-\vec{r}_i(t))$$

the total rate of work done is

$$\frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}(\vec{r}_i) = \int_V d^3r\, \vec{j}\cdot\vec{E}$$

So

$$\frac{d\mathcal{E}_{mech}}{dt} = \int_V d^3r\, \vec{j}\cdot\vec{E}$$

By maxwell equation $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$
we can write

$$\vec{j} = \frac{c}{4\pi}\left[\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right]$$

$$\int_V d^3r\, \vec{j}\cdot\vec{E} = \int_V d^3r\, \frac{c}{4\pi}\left[(\vec{\nabla}\times\vec{B})\cdot E - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}\right]$$

~~use~~ $\frac{\partial\vec{E}}{\partial t}\cdot E = \frac{1}{2}\frac{\partial E^2}{\partial t}$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot B - E\cdot(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

then use $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

$$\text{so } \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$= -\frac{1}{2c}\frac{\partial B^2}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Combine results to get

$$\int_V d^3r\ \vec{j}\cdot\vec{E} = -\frac{1}{4\pi}\int_V d^3r \left[\frac{1}{2}\frac{\partial B^2}{\partial t} + \frac{1}{2}\frac{\partial E^2}{\partial t} + c\,\vec{\nabla}\cdot(\vec{E}\times\vec{B})\right]$$

define $\boxed{u = \frac{1}{8\pi}(E^2+B^2)}$ electromagnetic energy density

$\boxed{\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}}$ Poynting vector - energy current

then

$$\frac{d\mathcal{E}_{mech}}{dt} = \int_V d^3r\,\vec{j}\cdot\vec{E} = -\int_V d^3r\left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S}\right]$$

If we define $\mathcal{E}_{EM}$, the electromagnetic energy of the volume V, as

$$\mathcal{E}_{EM} = \int_V d^3r\ u$$

then

$$\frac{d}{dt}\left(\mathcal{E}_{mech} + \mathcal{E}_{EM}\right) = -\oint_S da\ \hat{n}\cdot\vec{S}$$

~~or if we write $\frac{\partial u_{mech}}{\partial t} = \vec{j}\cdot\vec{E}$ as the time rate of change of mechanical~~

or we can write in differential form

$$\vec{j}\cdot\vec{E} + \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$$

local energy conservation law if interpret $\vec{S}$ as energy current and u as EM energy density

rate of change of mechanical energy per unit volume

$$\frac{d}{dt}\left(\mathcal{E}_{mech} + \mathcal{E}_{EM}\right) = -\oint_S da\, \hat{n}\cdot\vec{S}$$

total energy in V can decrease only if electromagnetic energy is being transported through the surface S by the EM energy current $\vec{S}$. assumes the charged particles do not leave the volume V.

under certain conditions, we can derive a similar conservation law for the macroscopic maxwell eqns.

Consider that $\vec{j}$ is current of the free $\overbrace{\text{particles}}^{\text{charged}}$.

Then repeating the above steps:

$$\int_V d\vec{r}^3\, \vec{j}\cdot\vec{E} = \frac{c}{4\pi}\int d^3r\ \vec{E}\cdot\left[\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$
$$= -\frac{1}{c}H\cdot\frac{\partial\vec{B}}{\partial t} - E\cdot\vec{\nabla}\times\vec{H}$$

so

$$\int_V d^3r\, \vec{j}\cdot\vec{E} = \frac{-1}{4\pi}\int_V d^3r\left[c\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right.$$

If the medium is linear, and we have quasistatic conditions, so that

$$\vec{D}(t) \simeq \epsilon\vec{E}(t)$$
$$\vec{H}(t) \simeq \frac{1}{\mu}\vec{B}(t)$$

then we can write

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \epsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{\mu}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2\mu}\frac{\partial B^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{H})$$

** Note: in general, as we will soon see, above conditions are not satisfied. ϵ will depend on frequency ω and $\vec{D}(t)$ and $\vec{E}(t)$ are non locally related in time. $\vec{D}(t) = \int_{-\infty}^{t} dt'\,\epsilon(t-t')\vec{E}(t')$. Only at low frequencies

ie in quasistatic case, can we write $\vec{D}(t) \simeq \epsilon(\omega=0)\vec{E}(t)$

Assume the above conditions are met, then

$$\int_V d^3r\,\vec{j}\cdot\vec{E} + \int_V d^3r\,\frac{\partial u}{\partial t} = -\oint_S da\,\hat{n}\cdot\vec{S}$$

where
$$\begin{cases} u = \frac{1}{8\pi}\left[\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right] \\ \vec{S} = \frac{c}{4\pi}\left[\vec{E}\times\vec{H}\right] \end{cases}$$

$\Rightarrow$ electromagnetic energy in dielectric + magnetic materials under quasi static conditions is

$$\int_V d^3r\left[\underbrace{\frac{1}{8\pi}\vec{E}\cdot\vec{D}}_{\text{electro static energy}} + \underbrace{\frac{1}{8\pi}\vec{B}\cdot\vec{H}}_{\text{magnetostatic energy}}\right]$$

Statics

Electrostatic Energy

Returning to microscopic fields and charges

$$\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, E^2 \qquad \text{use } \vec{E} = -\vec{\nabla}\phi$$

$$= \frac{-1}{8\pi}\int_V d^3r\, (\vec{\nabla}\phi)\cdot\vec{E} \qquad \text{use } \vec{\nabla}\cdot(\phi\vec{E}) = \phi\,\vec{\nabla}\cdot\vec{E} + (\vec{\nabla}\phi)\cdot\vec{E}$$

$$= \frac{-1}{8\pi}\int_V d^3r\left[\vec{\nabla}\cdot(\phi\vec{E}) - \phi\,\vec{\nabla}\cdot\vec{E}\right] \qquad \text{use } \vec{\nabla}\cdot E = 4\pi\rho$$

$$= \frac{1}{2}\int_V d^3r\, \rho\phi - \frac{1}{8\pi}\oint_S da\, \hat{n}\cdot\phi\vec{E} \qquad \text{by Gauss Theorem}$$

If let V be all space, $S\to\infty$, then $\phi \sim \frac{1}{r}$, $E \sim \frac{1}{r^2}$

surface integral $\sim \frac{R^2}{R^3} \to 0$ as $R\to\infty$.

$$\boxed{\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\,\phi}$$

can also use $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ to write

$$\boxed{\mathcal{E} = \frac{1}{2}\int d^3r\, d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

charge-charge interaction

Magnetostatic Energy

microscopic fields and currents

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\ B^2 \qquad \text{use } \vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \frac{1}{8\pi}\int d^3r\ \vec{B}\cdot\vec{\nabla}\times\vec{A} \qquad \text{use } \vec{\nabla}\cdot(\vec{B}\times\vec{A}) = \vec{A}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{A})$$

$$= \frac{1}{8\pi}\int d^3r\left[\vec{A}\cdot\vec{\nabla}\times\vec{B} - \vec{\nabla}\cdot(\vec{B}\times\vec{A})\right] \qquad \text{use } \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

$$= \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A} - \frac{1}{8\pi}\oint_S da\ \hat{n}\cdot(\vec{B}\times\vec{A})$$

as take V to fill all space, $S\to\infty$, surface term vanishes

$$\boxed{\mathcal{E} = \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A}}$$

In Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, $\vec{A}(\vec{r}) = \int d^3r'\ \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

In any other gauge we have $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ for some scalar χ. So we can always write

$$\vec{A}(\vec{r}) = \int d^3r'\ \frac{\vec{j}(\vec{r}\,')}{c\,|\vec{r}-\vec{r}\,'|} + \vec{\nabla}\chi$$

regardless of the choice of gauge, where χ is then determined so $\vec{A}$ satisfies the desired gauge condition

$$\mathcal{E} = \frac{1}{2c}\int d^3r\, d^3r'\, \frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{c\,|\vec{r}-\vec{r}'|} + \frac{1}{2c^2}\int d^3r\, \vec{j}\cdot\vec{\nabla}\chi$$

2nd term is $\int d^3r\, \vec{j}\cdot\nabla\chi = \int d^3r \left[\nabla\cdot(\vec{j}\chi) - \chi\,\vec{\nabla}\cdot\vec{j}\right]$

$$= \underbrace{\oint_S da\, \hat{n}\cdot\vec{j}\chi}_{\text{vanishes as } S\to\infty} - \underbrace{\int d^3r\, \chi\,\vec{\nabla}\cdot\vec{j}}_{\text{vanishes in magnetostatics where } \vec{\nabla}\cdot\vec{j}=0}$$

So

$$\boxed{\mathcal{E} = \frac{1}{2c^2}\int d^3r\, d^3r'\, \frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

current-current interaction

Momentum Conservation

For charges q_i at positions $\vec{r}_i$ with velocities $\dot{\vec{r}}_i$

$$\frac{d\vec{P}^{\,mech}}{dt} = \sum_i \vec{F}_i = \sum_i q_i\left(\vec{E}(\vec{r}_i) + \frac{1}{c}\vec{v}_i \times \vec{B}(\vec{r}_i)\right)$$

"mechanical" momentum of the charges

force on charge i

$$= \int_V d^3r \left[\rho\vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right]$$

$$\rho E + \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]$$

Now $\displaystyle \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) + \frac{1}{c}\left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$ use $\displaystyle \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

$$= \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) - \vec{E}\times(\vec{\nabla}\times\vec{E})$$

So $\displaystyle -\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\times\vec{B} = -\vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$

Therefore

$$\rho E + \frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}\Big[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\overbrace{(\vec{\nabla}\cdot\vec{B})}^{=0} - \vec{B}\times(\vec{\nabla}\times\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\Big]$$

Define electromagnetic momentum density

$$\boxed{\vec{\Pi} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S}}$$

($\vec{S}$ is Poynting vector)

then

$$\frac{d\vec{P}^{\,mech}}{dt} + \frac{d}{dt}\int_V d^3r\,\vec{\Pi} = \frac{1}{4\pi}\int_V d^3r\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + \vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

want to rewrite as a surface integral

ith component of integrand on right hand side is ($\vec{E}$ part only)
(sum over repeated indicies)

$$E_i \partial_j E_j - \varepsilon_{ijk} E_j \varepsilon_{klm} \partial_l E_m$$

$$= E_i \partial_j E_j - \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) E_j \partial_l E_m$$

$$= E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i$$

$$= \partial_j \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2\right)$$

Define <u>Maxwell's stress tensor</u>

$$\boxed{T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \frac{1}{2}\delta_{ij}\left(E^2 + B^2\right)\right]}$$

(note $T_{ij} = T_{ji}$ symmetric tensor)

Then

$$\frac{d}{dt} p_i^{mech} + \frac{d}{dt}\int_V d^3r\, \Pi_i = \int_V d^3r\, \partial_j T_{ij} \qquad \left(\partial_j T_{ij} = \frac{\partial T_{ij}}{\partial x_j}\right)$$

$$= \oint_S da\, T_{ij} \cdot \hat{n}_j$$

$$\frac{d}{dt}\vec{p}^{\,mech} + \frac{d}{dt}\int_V d^3r\, \vec{\Pi} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n}$$

- T_{ij} gives the flow of the i^{th} component of electromagnetic field momentum through an element of surface area $\perp$ ~~to~~ to direction $\hat{e}_j$

~~For static situations where~~ $\cancel{\frac{\partial \vec{\Pi}}{\partial t} = 0,\ \frac{d\vec{P}^{mech}}{dt} = \vec{F}^{tot} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n}}$
~~gives electromagnetic force on the surface S~~