Note: $\frac{d\vec{P}^{mech}}{dt}$ is ~~also~~ equal to the total electromagnetic force on the volume V.

Hence we can write

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T}\cdot\hat{n} - \frac{d}{dt}\int_V d^3r\, \vec{\Pi}$$

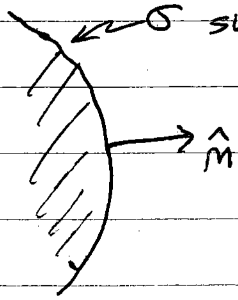
for <u>static</u> situations, the 2nd term vanishes and

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T}\cdot\hat{n}$$

$\overleftrightarrow{T}_{ij}$ is i^{th} component of static force on unit area with normal $\hat{e}_j$

this is origin of the term "stress" tensor.
$\overleftrightarrow{T}$ is like the stress tensor of an elastic medium.
T_{xx}, T_{yy}, T_{zz} are like pressure.
off diagonal elements are like shear stresses

Force on a conductor surface



net force on surface per unit area is

$$\vec{f} = \overleftrightarrow{T}^{\,above} \cdot \hat{n} - \overleftrightarrow{T}^{\,below} \cdot \hat{n}$$

$= 0$ as $\vec{E} = 0$ inside conductor

$$\vec{f} = \frac{1}{4\pi}\left[\vec{E}\,(\vec{E}\cdot\hat{n}) - \frac{1}{2}\hat{n}E^2\right]$$

for conducting surface

$$\hat{n}\cdot\vec{E}^{\,above} = 4\pi\sigma \quad (\text{since } \vec{E} \text{ below} = 0)$$

and tangential component $\vec{E} = 0$

$$\Rightarrow \vec{E} = 4\pi\sigma\,\hat{n}$$

So $\vec{f} = \frac{1}{4\pi}\left[(4\pi\sigma\,\hat{n})(4\pi\sigma) - \frac{1}{2}\hat{n}\,(4\pi\sigma)^2\right]$

$$\cancel{\vec{f} = 2\pi\sigma^2\hat{n}}$$

$$\vec{f} = \frac{\hat{n}}{4\pi}\left[(4\pi\sigma)^2 - \frac{1}{2}(4\pi\sigma)^2\right] = 2\pi\sigma^2\hat{n}$$

force per unit area: $\boxed{\vec{f} = 2\pi\sigma^2\hat{n} = \frac{1}{2}\sigma\vec{E}}$

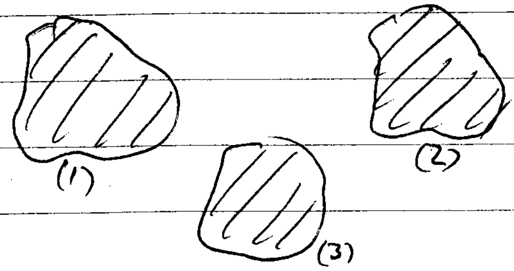
$\vec{f} = \sigma\vec{E}_{ave}$

where $\vec{E}_{ave} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below})$ is average field at surface averaging over above + below

Note factor ½. Naively one might have thought $\vec{f} = \sigma\vec{E}$. But need to exclude self field of charge on surface from acting on itself. See also Jackson pg 42 for another approach

Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor i



(also need condition on $\phi(\vec{r}) \to \infty$ if system is not enclosed)

From uniqueness theorem we know that specifying the V_i on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form.

Let $\phi^{(i)}(\vec{r})$ be the solution to the boundary value problem

$$\nabla^2 \phi^{(i)}(r) = 0 \quad \text{and} \quad \phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } (j),\ j \neq i \end{cases}$$

Then by superposition

$$\phi(\vec{r}) = \sum_i V_i \, \phi^{(i)}(\vec{r})$$

is solution to the problem $\nabla^2 \phi = 0$ and $\phi(\vec{r}) = V_i$ for $\vec{r}$ on surface of conductor (i)

The surface charge density at $\vec{r}$ on surface of conductor (i) is

$$\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi} \frac{\partial \phi(\vec{r})}{\partial n} = -\frac{1}{4\pi} \sum_j V_j \frac{\partial \phi^{(j)}(\vec{r})}{\partial n}$$

where $\dfrac{\partial \phi}{\partial n} = (\vec{\nabla} \phi) \cdot \hat{n}$ is the derivative normal to the surface at point $\vec{r}$.

The total charge on conductor (i) is

$$Q_i = \int_{S_i} da\, \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi}\sum_j V_j \int_{S_i} da\, \frac{\partial \phi^{(j)}}{\partial n}$$

surface of conductor (i)

Define $C_{ij} \equiv -\frac{1}{4\pi}\int_{S_i} da\, \frac{\partial \phi^{(j)}}{\partial n}$ the C_{ij} depend only on the geometry of the conductors

Then we have

$$\boxed{Q_i = \sum_j C_{ij} V_j}$$

C_{ij} is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials V_j on the conductors (j)

Since we know that specifying the Q_i that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential V_i on each conductor, the capacitance matrix is invertable

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\, \phi = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j$$

Common to define capacitance of two conductors by

$$C = \frac{Q}{V_1 - V_2}$$

when conductor (1) has charge Q
conductor (2) has charge $-Q$
$V_1 - V_2$ is potential difference between the two conductors.
all other conductors fixed at $V_i = 0$

We can determine C in terms of the elements of the matrix C_{ij}

$$\left.\begin{aligned} Q &= C_{11} V_1 + C_{12} V_2 \\ -Q &= C_{21} V_1 + C_{22} V_2 \end{aligned}\right\} \Rightarrow V_2 = -\left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}}\right) V_1$$

$$\Rightarrow Q = \left[C_{11} - C_{12} \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1$$

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12} \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}{1 + \left(\frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}$$

$$\boxed{C = \frac{C_{11} C_{22} - C_{12} C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}}$$

Capacitance can also be defined when the space between the conductors is filled with a dielectric ϵ. In this case, if Q_i is the free charge, then Q_i / ϵ is the effective total charge to use in computing ϕ.

$$\Rightarrow \frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j$$

where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors

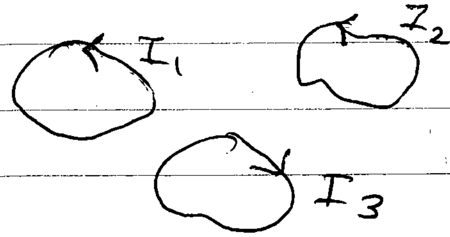
$$\Rightarrow Q_i = \sum_j \epsilon C_{ij}^{(0)} V_j$$

$$= \sum_j C_{ij} V_j \quad \text{where } C_{ij} = \epsilon C_{ij}^{(0)}$$

the capacitance is increased by a factor the dielectric constant ϵ.

Inductance

Consider a set of current carrying loops C_i with currents I_i



In Coulomb gauge, we can write the magnetic vector potential $\vec{A}$ from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

↑ integrate over loop C_i
integration variable is $\vec{r}'$

The magnetic flux through loop i is

$$\Phi_i = \int_{S_i} da\, \hat{n}\cdot\vec{B} = \int_{S_i} da\, \hat{n}\cdot\vec{\nabla}\times\vec{A} = \oint_{C_i} d\vec{l}\cdot\vec{A}$$

↑ surface bounded by loop C_i

$$\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i}\oint_{C_j} \frac{d\vec{l}\cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

pure geometrical quantity

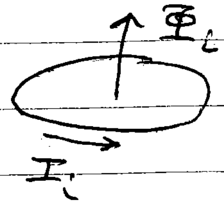
$$\boxed{\Phi_i \equiv c\sum_j M_{ij} I_j}$$

where $M_{ij} = \oint_i \oint_j \frac{d\vec{l}_i\cdot d\vec{l}_j}{c^2|\vec{r}-\vec{r}'|}$

is the mutual inductance of loops (i) and (j). $M_{ij} = M_{ji}$

$L_i \equiv M_{ii}$ is <u>self-inductance</u> of loop (i)

The sign convention in the above is that, Φ_i is computed in direction given by right hand rule, according to the direction taken for current in loop (i)



Magnetostatic energy

$$\mathcal{E} = \frac{1}{2c}\int d^3r \; \vec{j}\cdot\vec{A} = \frac{1}{2c}\sum_i \oint_{C_i} d\vec{l}\cdot\vec{A}\; I_i$$

$$= \frac{1}{2c}\sum_i \Phi_i I_i$$

$$\mathcal{E} = \frac{1}{2}\sum_{i,j} M_{ij} I_i I_j$$

Force and torque on electric dipoles

localized charge distribution $\rho(\vec{r})$ with net charge $\int d^3r\, \rho = 0$

force on ρ in slowly varying electric field $\vec{E}$ is

$$\vec{F} = \int d^3r\, \rho(\vec{r})\, \vec{E}(\vec{r})$$

define $\vec{r} = \vec{r}_0 + \vec{r}\,'$ where $\vec{r}_0$ is some fixed reference point in center of charge distr ρ, and $\vec{r}\,'$ is distance relative to $\vec{r}_0$

$$\vec{F} = \int d^3r'\, \rho(\vec{r}\,')\, \vec{E}(\vec{r}_0 + \vec{r}\,')$$

since $\vec{E}$ is slowly varying on length scale where $\rho \neq 0$, we expand

$$\vec{F} \cong \int d^3r'\, \rho(\vec{r}\,') \left[\vec{E}(\vec{r}_0) + (\vec{r}\,' \cdot \vec{\nabla}) \vec{E}(\vec{r}_0) \right] + \cdots$$

$$= \vec{E}(\vec{r}_0) \int d^3r'\, \rho(\vec{r}\,') + \left(\int d^3r'\, \rho(r')\, \vec{r}\,' \cdot \vec{\nabla} \right) \vec{E}(\vec{r}_0)$$

$$= 0 + (\vec{p} \cdot \vec{\nabla}) \vec{E}(\vec{r}_0)$$

$$\boxed{\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = \sum_{\alpha=1}^{3} p_\alpha \frac{\partial \vec{E}}{\partial r_\alpha}}$$

For $\vec{E} = $ constant, $\vec{F} = 0$

Torque on ρ is ~~[illegible]~~

$$\vec{N} = \int d^3r\ \rho(\vec{r})\ \vec{r} \times \vec{E}(\vec{r}) \cong \int d^3r\ \rho(\vec{r})\ \vec{r} \times \left[\vec{E}(\vec{r}_0) + \cdots\right]$$

to lowest order $\boxed{\vec{N} = \vec{p} \times \vec{E}}$

Force and torque on magnetic dipoles

localized magnetostatic current distribution $\vec{j}(\vec{r})$

$$\vec{F} = \frac{1}{c}\int d^3r\ \vec{j} \times \vec{B}$$

expand about center of current $\vec{r}_0$

$$\vec{B}(\vec{r}) \simeq \vec{B}(\vec{r}_0) + (\vec{r}' \cdot \vec{\nabla})\vec{B}(\vec{r}_0) + \cdots$$

$$\vec{F} = \frac{1}{c}\left[\int d^3r'\ \vec{j}(\vec{r}')\right] \times \vec{B}(\vec{r}_0) + \frac{1}{c}\int d^3r'\ \vec{j}(\vec{r}') \times (\vec{r}' \cdot \vec{\nabla})\vec{B}(\vec{r}_0)$$

from discussion of magnetic dipole approx we had $\int d^3r\ \vec{j} = 0$ for magnetostatics where $\vec{\nabla} \cdot \vec{j} = 0$, so 1st term vanishes. The 2nd term can be written as

$$F_\alpha = \frac{\varepsilon_{\alpha\beta\gamma}}{c}\int d^3r'\ j_\beta\ r'_\delta\ \partial_\delta B_\gamma$$

for magnetostatics see magnetic dipole derivation

we need the tensor $\frac{1}{c}\int d^3r'\ j_\beta\ r'_\delta = -\frac{1}{c}\int d^3r'\ r'_\beta\ j_\delta$

$$= \frac{1}{2c}\int d^3r' \left[j_\beta r'_\delta - r'_\beta j_\delta \right]$$

$$= -m_\sigma\ \varepsilon_{\sigma\beta\delta}$$

magnetic dipole $\vec{m} = \frac{1}{2c}\int d^3r\ \vec{r} \times \vec{j}$

$$F_\alpha = \varepsilon_{\alpha\beta\gamma}\,\varepsilon_{\sigma\beta\delta}(-m_\sigma)\,\partial_\delta B_\gamma$$

$$= -\left(\delta_{\alpha\sigma}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\sigma\gamma}\right) m_\sigma\,\partial_\delta B_\gamma$$

$$= \vec{\nabla}_\alpha(\vec{m}\cdot\vec{B}) - \vec{m}_\alpha\,\vec{\nabla}\cdot\vec{B}$$

$$\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})} \qquad \text{as } \vec{\nabla}\cdot\vec{B} = 0$$

torque on $\vec{j}$ is

$$\vec{N} = \frac{1}{c}\int d^3r\; \vec{r}\times(\vec{j}\times\vec{B})$$

to lowest order, $\vec{B} = \vec{B}(\vec{r}_0)$ is const over region where $\vec{j} \neq 0$

$$= \frac{1}{c}\int d^3r\left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{B}\,(\vec{r}\cdot\vec{j})\right]$$

2nd term $= 0$ as follows

$$\int d^3r\;\vec{r}\cdot\vec{j} = \int d^3r\;\vec{j}\cdot\vec{\nabla}\left(\frac{r^2}{2}\right) \qquad \text{as } \vec{\nabla}\left(\frac{r^2}{2}\right) = \vec{r}$$

$$= -\int d^3r\,(\vec{\nabla}\cdot\vec{j})\left(\frac{r^2}{2}\right)$$

integrate by parts. Surface term $\to 0$ as $\vec{j}$ is localized

$$= 0 \qquad \text{as } \vec{\nabla}\cdot\vec{j} = 0 \text{ in magnetostatics}$$

1st term involves

see derivation of magnetic dipole approx

$$\int d^3r\;\vec{j}\,\vec{r} = -\int d^3r\;\vec{r}\,\vec{j} = \frac{1}{2}\int d^3r\left[\vec{j}\,\vec{r} - \vec{r}\,\vec{j}\right]$$

So

$$\vec{N} = \frac{1}{2c}\int d^3r\left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B})\right]$$

$$\vec{N} = \frac{1}{2c} \int d^3r \left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B}) \right]$$

$$\cancel{= \sim \vec{m} \times \vec{B}}$$

$$= \frac{1}{2c} \int d^3r \,(\vec{r} \times \vec{j}) \times \vec{B}$$

$$\boxed{\vec{N} = \vec{m} \times \vec{B}}$$