

EM waves in Conductors

Conduction electrons are mobile, not bound
 $\Rightarrow$ we have to include the $\vec{j}_f$ and ρ_f from them.

Simple Classical model for electron motion - "Drude" Model

$$m \ddot{\vec{r}} = -e \vec{E}(t) - \frac{m}{\tau} \dot{\vec{r}}$$

$\uparrow$
external
E field

$\uparrow$
damping force due to collisions
 τ is "relaxation time"

$$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t} \text{ solution}$$

plug in to get

$$(-\omega^2 - \frac{i\omega}{\tau}) \vec{r}_\omega = -\frac{e}{m} \vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m} \frac{1}{\omega^2 + \frac{i\omega}{\tau}} \vec{E}_\omega$$

$$\text{current is } \vec{j}_f = -en \dot{\vec{r}}_\omega = -en(-i\omega) \vec{r}_\omega$$

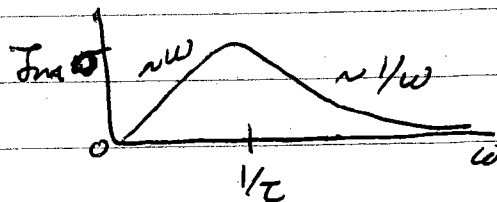
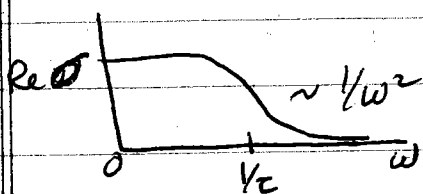
$\uparrow$ density of electrons

$$\vec{j}_f = \frac{ne^2}{m} \frac{i\omega}{\omega^2 + \frac{i\omega}{\tau}} \vec{E}_\omega = \frac{me^2\tau}{m} \frac{1}{1 - i\omega\tau} \vec{E}_\omega$$

$$\vec{j}_f = \sigma(\omega) \vec{E}_\omega$$

conductivity

$$\sigma(\omega) = \frac{me^2\tau}{m} \frac{1}{1 - i\omega\tau}$$



$$\text{Re } \sigma = \frac{\sigma_0}{1 + \omega^2 \tau^2}$$

$$\text{Im } \sigma = \frac{\sigma_0 \omega \tau}{1 + \omega^2 \tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{ne^2 \tau}{m}$$

dc conductivity

Charge density ρ_f given by charge conservation law,
for plane waves

$$\rho_f = \rho_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad \vec{j}_f = \vec{j}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{j}_f \Rightarrow -i\omega \rho_\omega = -i\vec{k} \cdot \vec{j}_\omega$$

$$\rho_\omega = \frac{\vec{k} \cdot \vec{j}_\omega}{\omega} = \frac{\sigma(\omega) \vec{k} \cdot \vec{E}_\omega}{\omega}$$

Maxwell Equations

$$1) \quad \vec{\nabla} \cdot \vec{D} = 4\pi \rho_f$$

$$2) \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$3) \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$4) \quad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j}_f + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

Assume $\vec{H} = \vec{B}/\mu$, μ constant

$$\vec{D}_\omega = \epsilon_b(\omega) \vec{E}_\omega$$

$\epsilon_b(\omega)$ is dielectric function
from the bound charges

$$\vec{j}_\omega = \sigma(\omega) \vec{E}_\omega$$

$\sigma(\omega)$ is conductivity from
free charges

$$\rho_\omega = \frac{\sigma}{\omega} \vec{k} \cdot \vec{E}_\omega$$

For harmonic plane wave solutions $\vec{E} = E_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$
etc.

$$1) \Rightarrow i\vec{k} \cdot \vec{D}_\omega = i\vec{k} \cdot \epsilon_b \vec{E}_\omega = 4\pi f_\omega = 4\pi\sigma \frac{\vec{k} \cdot \vec{E}_\omega}{\omega}$$

$$\Rightarrow i\vec{k} \cdot \vec{E}_\omega \left(\epsilon_b + \frac{4\pi i\sigma}{\omega} \right) = 0$$

$$2) \Rightarrow i\mu \vec{k} \cdot \vec{H}_\omega = 0$$

$$3) \Rightarrow i\vec{k} \times \vec{E}_\omega = \frac{i\omega}{c} \vec{B}_\omega = \frac{i\omega\mu}{c} \vec{H}_\omega$$

$$\begin{aligned} 4) \Rightarrow i\vec{k} \times \vec{H}_\omega &= \frac{4\pi}{c} \vec{J}_\omega - \frac{i\omega}{c} \vec{D}_\omega \\ &= \frac{4\pi\sigma}{c} \vec{E}_\omega - \frac{i\omega}{c} \epsilon_b \vec{E}_\omega \\ &= -\frac{i\omega}{c} \left(\epsilon_b + \frac{4\pi i\sigma}{\omega} \right) \vec{E}_\omega \end{aligned}$$

Notice: all the equations above look exactly like what we had for the dielectric, provided we define

$$\epsilon(\omega) = \epsilon_b(\omega) + \frac{4\pi i\sigma(\omega)}{\omega}$$

So all results for the dielectric case carry over to conductors, provided we make the above substitution. In particular

dispersion relation
for transverse modes

$$k^2 = \frac{\omega^2}{c^2} \epsilon(\omega)$$

The main difference between dielectrics & conductors has to do with the contribution that the $4\pi i\sigma/\omega$ makes to the real and imaginary parts of $\epsilon(\omega)$.

For single Drude model $\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$ $\sigma_0 = \frac{ne^2\tau}{m}$

① Low frequencies $\omega \ll 1/\tau$

$$\epsilon_b(\omega) \approx \epsilon_b(0) \text{ real}$$

$$\sigma(\omega) \approx \sigma_0 \text{ real}$$

$$\Rightarrow \boxed{\epsilon(\omega) \approx \epsilon_b(0) + \frac{4\pi i\sigma_0}{\omega}}$$

← gives large ϵ_2 as $\omega \rightarrow 0$

⇒ strong dissipation

$$\text{Re } \epsilon = \epsilon_1$$

$$\text{Im } \epsilon = \epsilon_2$$

$$\text{when } \frac{\epsilon_2}{\epsilon_1} = \frac{4\pi\sigma_0}{\omega\epsilon_b(0)} \gg 1$$

we call this regime a "good" conductor.

Conduction electrons dominate the response
— waves strongly attenuated

$$\text{when } \frac{\epsilon_2}{\epsilon_1} = \frac{4\pi\sigma_0}{\omega\epsilon_b(0)} \ll 1$$

we call this regime a "poor" conductor.

little absorption of energy by conduction electrons.

waves propagate

one always enters the "good" conductor region when ω gets sufficiently small.

wave vector:

$$k = \frac{\omega}{c} \sqrt{\mu \epsilon}$$

for a good conductor where $\epsilon_2 \gg \epsilon_1$,

$$\epsilon \sim i\epsilon_2 = \frac{4\pi i\sigma_0}{\omega}$$

$$k = k_1 + ik_2 = \frac{\omega}{c} \sqrt{\mu \frac{4\pi i\sigma_0}{\omega}} \quad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k_1 = k_2 = \frac{\omega}{c} \sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c} \sqrt{2\pi\mu\sigma_0\omega}$$

for $\vec{k} = k\hat{z}$,

$$\vec{E} = \vec{E}_\omega e^{i(kz - \omega t)} = \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

$$\delta \equiv 1/k_2 = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}} \quad \begin{array}{l} \text{"skin depth"} \\ \text{distance wave} \\ \text{propagates into} \\ \text{conductor} \end{array}$$

$$\delta \sim 1/\sqrt{\omega} \quad \begin{array}{l} \text{increases as} \\ \omega \text{ decreases} \end{array}$$

ϕ phase shift between oscillations of $\vec{E}$ and $\vec{H}$

$$\phi = \arctan(k_2/k_1) \approx \arctan(1) = 45^\circ$$

$$\text{Amplitude ratio } \frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}c}{\omega\mu} k_1$$

$$= \frac{\sqrt{2}c}{\omega\mu} \frac{1}{c} \sqrt{2\pi\mu\sigma_0\omega}$$

$$= \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim 1/\sqrt{\omega}$$

as $\omega \rightarrow 0$, most of the energy of the wave is carried by the magnetic field part

② high frequencies $\omega \gg 1/\tau$, $\omega \gg \omega_0$

$$\epsilon_b(\omega) \approx 1$$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{ime^2\tau}{m\omega\tau} = \frac{ime^2}{m\omega}$$

pure ima
dep of
 τ

$$\epsilon(\omega) \approx 1 + \frac{4\pi i\sigma}{\omega} \approx 1 - \frac{4\pi me^2}{m\omega^2}$$

$$\boxed{\epsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}}$$

$$\omega_p = \sqrt{\frac{4\pi me^2}{m}}$$

plasma freq of the
conduction electrons

$\epsilon(\omega)$ is real

1) If $\omega > \omega_p$ then $\epsilon > 0$

$\Rightarrow$ transparent propagation

$$k = k_1 = \frac{\omega}{c} \sqrt{\mu\epsilon} \text{ is pure real}$$

$$k_2 \approx 0$$

2) If $\omega < \omega_p$ then $\epsilon < 0$

$\Rightarrow$ total reflection

$$k_1 \approx 0$$

k is pure imag

$$k = k_2 = \frac{\omega}{c} \sqrt{\mu|\epsilon|}$$

ω_p gives cross over between total reflection
and transparent propagation

for typical metals

$$\tau \sim 10^{-14} \text{ sec}$$

$$\omega_p \sim 10^{16} \text{ sec}^{-1}$$

$$\lambda_p = \frac{2\pi c}{\omega_p} \sim 3 \times 10^3 \text{ \AA} \quad (\text{visible is } \lambda \sim 5 \times 10^3 \text{ \AA})$$

Example: The ionosphere is a layer of charged gas surrounding the earth. In many respects the charged particles of the ionosphere behave like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio signals reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio signals propagate through ionosphere into space

Explains why you can pick up AM stations from far away - they get reflected back. But you can only pick up local FM stations.

Longitudinal modes in conductors

ie $\vec{H}_\omega$ or $\vec{E}_\omega$ not $\perp \vec{k}$
magnetic field

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow i\mu \vec{k} \cdot \vec{H}_\omega = 0 \Rightarrow \vec{H}_\omega \perp \vec{k} \text{ transverse}$$

or $\vec{k} = 0$ spatially uniform $\vec{H}$

if $\vec{k} = 0$ then Faraday

$$i\vec{k} \times \vec{E}_\omega = i\omega\mu \vec{H}_\omega = 0 \Rightarrow \omega = 0$$

" as $\vec{k} = 0$

So only possible longitudinal $\vec{H}$ is
spatially uniform, constant in time.

electric field

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f \Rightarrow i\varepsilon(\omega) \vec{k} \cdot \vec{E}_\omega = 0 \Rightarrow \vec{E}_\omega \perp \vec{k} \text{ transverse}$$

or $\varepsilon(\omega) = 0$

If $\vec{E}_\omega \parallel \vec{k}$ but $\varepsilon(\omega) = 0$, then can satisfy all
other maxwell equations.

$$i\vec{k} \times \vec{E}_\omega = \frac{i\omega\mu}{c} \vec{H}_\omega \Rightarrow \vec{H}_\omega = 0$$

$$\Rightarrow i\mu \vec{k} \cdot \vec{H}_\omega = 0 \text{ and } i\vec{k} \times \vec{H}_\omega = -\frac{i\omega\varepsilon(\omega)}{c} \vec{E}_\omega$$

" as $\vec{H}_\omega = 0$ " as $\varepsilon(\omega) = 0$

So we can have longitudinal electric field oscillation
when $\varepsilon(\omega) = 0$

low freq $\omega \ll \omega_0$ $\omega \tau \ll 1$

$$\epsilon \approx \epsilon_b(0) + \frac{4\pi i \sigma_0}{\omega}$$

$$\epsilon(\omega) = 0 \quad \text{when} \quad \omega = -\frac{4\pi i \sigma_0}{\epsilon_b(0)}$$

$$\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_\omega e^{-\frac{4\pi \sigma_0}{\epsilon_b(0)} t} e^{i\vec{k} \cdot \vec{r}}$$

If set up a longitudinal $\vec{E}$ field, it decays to zero exponentially with ~~time~~ decay time $\epsilon_b(0)/4\pi\sigma_0$. This is consistent with assumption the $\vec{E}=0$ inside a conductor for electrostatics.

in statics $\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{E} \sim -i\vec{k}\phi_k e^{i\vec{k} \cdot \vec{r}}$ is longitudinal

high freq $\omega \gg 1/\tau$, $\omega \gg \omega_0$

$$\epsilon(\omega) \approx 1 + \frac{4\pi e^2 N}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \quad \omega_p^2 = \frac{4\pi m e^2 N}{m}$$

$$\epsilon = 0 \quad \text{when} \quad \omega = \omega_p$$

So we have oscillatory longitudinal $\vec{E}$ only when $\omega = \omega_p$, independent of $\vec{k}$.

$$\vec{E} = \vec{E}_\omega e^{i\vec{k} \cdot \vec{r}} e^{-i\omega_p t}$$

This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a plasmon.

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \Rightarrow \rho = \frac{i\vec{k} \cdot \vec{E}_\omega}{4\pi} e^{i\vec{k} \cdot \vec{r}} e^{-i\omega_p t} \quad \left\{ \begin{array}{l} \text{plasma osc.} \\ \text{is a charge} \\ \text{density oscillation} \end{array} \right.$$