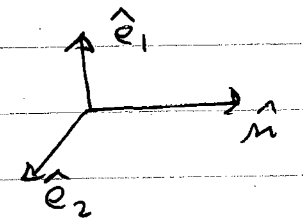
Polarization

Consider transverse wave propagating in direction $\hat{n}$ ie $\vec{k} = k\hat{n}$.



$$\hat{e}_1 \times \hat{e}_2 = \hat{n}$$
$$\hat{n} \times \hat{e}_1 = \hat{e}_2$$
$$\hat{e}_2 \times \hat{n} = \hat{e}_1$$

A general solution for a transverse wave has the form

$$\vec{E}(\vec{r},t) = Re\left\{(E_1\hat{e}_1 + E_2\hat{e}_2)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$$

$$\vec{H}(\vec{r},t) = \frac{c}{\omega\mu} Re\left\{k\,\hat{n}\times(E_1\hat{e}_1 + E_2\hat{e}_2)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$$

$$= \frac{c}{\mu\omega} Re\left\{k(E_1\hat{e}_2 - E_2\hat{e}_1)e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$$

So far we considered implicitly only the case

E_1, E_2 real constants $\rightarrow |\vec{E}_\omega| = \sqrt{E_1^2+E_2^2}$

But most general case is

$$E_1 = E\cos\theta\, e^{i\chi_1}$$
$$E_2 = E\sin\theta\, e^{i\chi_2}$$

} need not be real, can be complex with relative phase difference

$$E^2 = |E_1|^2 + |E_2|^2$$

(for $\chi_1 = \chi_2 = 0$, θ is angle $\vec{E}_\omega$ makes with respect to $\hat{e}_1$,

define $\Phi = k_1\hat{n}\cdot\vec{r} - \omega t$

$\tan\varphi = k_2/k_1$

$$\vec{E} = E\, e^{-k_2 \hat{n}\cdot\vec{r}} \left[\hat{e}_1 \cos\theta \cos(\Phi + \chi_1) + \hat{e}_2 \sin\theta \cos(\Phi + \chi_2) \right]$$

$$\vec{H} = \frac{c|k|}{\omega\mu} E\, e^{-k_2 \hat{n}\cdot\vec{r}} \left[\hat{e}_2 \cos\theta \cos(\Phi + \chi_1 + \varphi) - \hat{e}_1 \sin\theta \cos(\Phi + \chi_2 + \varphi) \right]$$

special cases

1) $\underline{\chi_1 = \chi_2}$

$$\vec{E} = (\hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta) E\, e^{-k_2 \hat{n}\cdot\vec{r}} \cos(\Phi + \chi_1)$$
$$\vec{H} = (-\hat{e}_1 \sin\theta + \hat{e}_2 \cos\theta) \frac{|k|}{\omega\mu} E\, e^{-k_2 \hat{n}\cdot\vec{r}} \cos(\Phi + \chi_1 + \varphi)$$

Linearly polarized - $\vec{E}$ and $\vec{H}$ point in fixed directions and are orthogonal. phase shift is φ

2) $\underline{\cos\theta = \frac{1}{\sqrt{2}},\ \sin\theta = \pm 1/\sqrt{2}}$, $\chi_1 = 0$, $\chi_2 = \chi$ (can always choose $\chi_1 = 0$ by shifting the time scale)

$$\vec{E}^{\pm} = \frac{E}{\sqrt{2}}\left(\hat{e}_1 \cos\Phi \pm \hat{e}_2 \cos(\Phi + \chi)\right)$$

Find locus of points that $\vec{E}$ sweeps out as Φ varies.

$$\vec{E}^{\pm}\cdot\hat{e}_1 = E_1^{\pm} = \frac{E}{\sqrt{2}} \cos\Phi$$
$$\vec{E}^{\pm}\cdot\hat{e}_2 = E_2^{\pm} = \pm\frac{E}{\sqrt{2}} \cos(\Phi + \chi)$$

Define $\Theta = \Phi + \chi/2$ so

$$E_1^{\pm} = \frac{E}{\sqrt{2}} \cos(\Theta - \chi/2)$$
$$E_2^{\pm} = \pm\frac{E}{\sqrt{2}} \cos(\Theta + \chi/2)$$

$$E_1^{\pm} = \frac{E}{\sqrt{2}} \cos\Theta \cos\chi/2 + \frac{E}{\sqrt{2}} \sin\Theta \sin\chi/2$$
$$E_2^{\pm} = \pm\frac{E}{\sqrt{2}} \cos\Theta \cos\chi/2 \mp \frac{E}{\sqrt{2}} \sin\Theta \sin\chi/2$$

$$\Rightarrow \quad E_1^+ + E_2^+ = \frac{2E}{\sqrt{2}} \cos \Phi \cos \chi/2$$
$$E_1^+ - E_2^+ = \frac{2E}{\sqrt{2}} \sin \Phi \sin \chi/2$$

$$\Rightarrow \quad \frac{(E_1^+ + E_2^+)^2}{2E^2 \cos^2 \chi/2} + \frac{(E_1^+ - E_2^+)^2}{2E^2 \sin^2 \chi/2} = \cos^2 \Phi + \sin^2 \Phi = 1$$

<u>Similarly</u>

$$\frac{(E_1^- - E_2^-)^2}{2E^2 \cos^2 \chi/2} + \frac{(E_1^- + E_2^-)^2}{2E^2 \sin^2 \chi/2} = 1$$

These are the equations for <u>ellipses</u>! $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

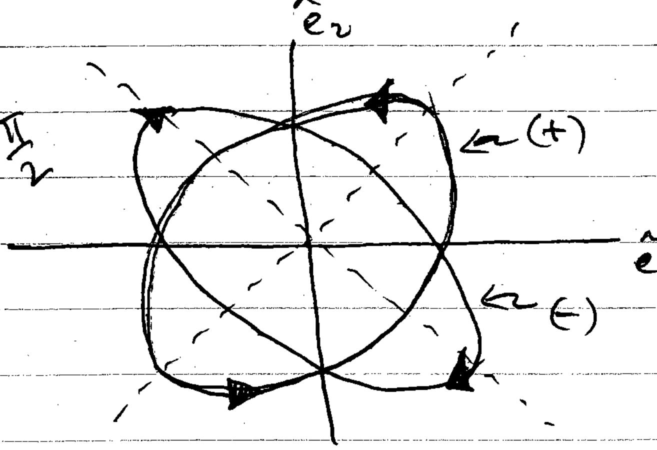
with semi axes

$E \cos \chi/2$ and $E \sin \chi/2$ $\quad \hat{e}_1{}' \quad \hat{e}_2{}'$

direction of the ellipse axes are $\left(\frac{\hat{e}_1 + \hat{e}_2}{\sqrt{2}}\right)$ and $\left(\frac{\hat{e}_1 - \hat{e}_2}{\sqrt{2}}\right)$

$\vec{E} \cdot \hat{e}_1{}' = \frac{E_1 + E_2}{\sqrt{2}}$, $\vec{E} \cdot \hat{e}_2{}' = \frac{E_1 - E_2}{\sqrt{2}}$

for $|\chi| < \frac{\pi}{2}$



so $|\vec{E}|$ sweeps out ellipse as Φ varies.

$\Rightarrow$ sit at position $\vec{r}$. as t varies, $|\vec{E}|$ sweeps out ellipse axis at 45° to $\hat{e}_1, \hat{e}_2$

<u>elliptically polarized wave</u>

For $0 < \chi < \pi/2$

For (+) $\vec{E}$ moves around ellipse counterclockwise (right-handed)

For (−) $\vec{E}$ moves around ellipse clockwise (left handed)

as t increases (ie as Φ decreases)

3) <u>special case of 1c) $\chi = \pi/2$</u>

$$\cos^2 \chi/2 = \sin^2 \chi/2 = \frac{1}{2}$$ ellipse axes are equal!

$\Rightarrow \vec{E}$ sweeps out a circular path

<u>circularly polarized waves</u>

(+) goes counterclockwise

(−) goes clockwise

One defines <u>circular polarization basis</u> vectors as:

$$\hat{e}_\pm = \frac{1}{\sqrt{2}}(\hat{e}_1 \pm i\hat{e}_2)$$

$$\Rightarrow \quad \hat{e}_\pm^* \cdot \hat{e}_\pm = \hat{e}_\pm \cdot \hat{e}_\mp = 1$$

$$\hat{e}_\pm \cdot \hat{e}_\mp^* = \hat{e}_\pm \cdot \hat{e}_\pm = 0$$

$$\hat{e}_\pm \cdot \hat{n} = 0$$

$$\hat{n} \times \hat{e}_\pm = i\hat{e}_\pm^*$$

with this notation, a circularly polarized wave is

$$\vec{E} = Re\left\{ E\hat{e}_\pm e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right\}$$

(+) is counterclock.
(−) is clockwise
with $\hat{n}$ pointing <u>out</u>

Note:

$$\frac{1}{\sqrt{2}}\left(E\,\hat{e}_+ + E\,\hat{e}_-\right) = \frac{E}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\hat{e}_1 + \frac{i}{\sqrt{2}}\hat{e}_2 + \frac{1}{\sqrt{2}}\hat{e}_1 - \frac{i}{\sqrt{2}}\hat{e}\right.$$

$$= E\hat{e}_1$$

and $\frac{1}{\sqrt{2}i}(E\hat{e}_+ - E\hat{e}_-) = E\hat{e}_2$

Thus a linearly polarized wave can be written as a superposition of counter rotating circularly polarized waves!

general case $\quad E_1, E_2$ complex

write $E_1\hat{e}_1 + E_2\hat{e}_2 \equiv \vec{U} e^{i\psi}$

where ψ is chosen so that $\vec{U}\cdot\vec{U}$ is real

(can always do this since $\vec{U}\cdot\vec{U} = (E_1^2+E_2^2)e^{-2i\psi}$

so 2ψ is just the phase of $E_1^2+E_2^2$)

$\vec{U}$ is complex vector $\Rightarrow \vec{U} = \vec{U}_a - i\vec{U}_b$, $\vec{U}_a, \vec{U}_b$ real

since $\vec{U}\cdot\vec{U}$ is real $\Rightarrow \vec{U}_a\cdot\vec{U}_b = 0$

so $\vec{U}_a \perp \vec{U}_b$ orthogonal

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{ \vec{U} e^{i\psi} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\} \qquad \Phi \equiv k_1\hat{n}\cdot\vec{r} - \omega t$$

$$= e^{-k_2\hat{n}\cdot\vec{r}} \left\{ \vec{U}_a \cos(\Phi+\psi) + \vec{U}_b \sin(\Phi+\psi) \right\}$$

Define E_a as component of $\vec{E}$ in direction of $\vec{U}_a$

E_b as component of $\vec{E}$ in direction of $\vec{U}_b$

$$E_a = U_a \cos(\Phi+\psi)$$
$$E_b = U_b \sin(\Phi+\psi)$$

(ignore attenuation factor by either absorbing it into U_a, U_b, or consider $\vec{r}=0$)

$$\Rightarrow \left(\frac{E_a}{U_a}\right)^2 + \left(\frac{E_b}{U_b}\right)^2 = 1$$

elliptical polarization

$U_a \equiv |\vec{U}_a|$

$U_b \equiv |\vec{U}_b|$

semi axes of lengths U_a and U_b oriented in directions $\vec{U}_a$ and $\vec{U}_b$

if $U_a = \pm U_b$ then circularly polarized

Define circular polarization basis vectors

$$\hat{e}_\pm = \frac{1}{\sqrt{2}}(\hat{e}_a \pm i\hat{e}_b) \qquad \hat{e}_a = \frac{\vec{U}_a}{|\vec{U}_a|}, \quad \hat{e}_b = \frac{\vec{U}_b}{|\vec{U}_b|}$$

any general $\vec{U}$ can always be written as:

$$\vec{U} = \frac{1}{\sqrt{2}}(U_a + U_b)\,\hat{e}_- + \frac{1}{\sqrt{2}}(U_a - U_b)\,\hat{e}_+$$

Thus a general elliptically polarized wave can be written as a superposition of circularly polarized waves!

<u>Consider behavior of magnetic field</u>

for $\chi_1 = 0$, $\chi_2 = \chi$

$$\vec{E}\cdot\vec{H} = \frac{c|k|}{\omega\mu}E^2 e^{-2k_2\hat{n}\cdot\vec{r}}\cos\theta\sin\theta\left[\cos(\Phi+\chi)\cos(\Phi+\varphi) - \cos(\Phi)\cos(\Phi+\chi+\varphi)\right]$$

can show that $[\cdots] = \sin\varphi\sin\chi$

use your trig identities!

$\Rightarrow$ $\vec{H} \perp \vec{E}$ when (i) $\chi = 0$ ie linear polarization for any φ, ie any dissipation

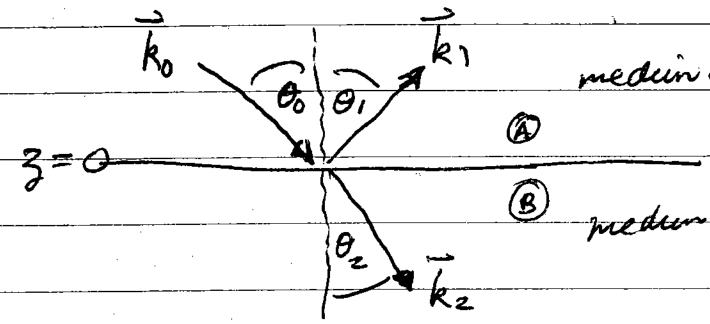
or (ii) $\varphi = 0$, ie no dissipation, for any χ, ie any non-linear polarization

$\vec{E}\cdot\vec{H}$ is independent of Φ

$\Rightarrow$ it is constant in time and space

for elliptically polarized wave in dissipative medium, $\vec{E}\cdot\vec{H} \neq 0$

Reflection + Transmission of Waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume ϵ_a is real and positive, ϵ_b may be complex
μ_a and μ_b are real and constant

$\vec{k}_0$ is incident wave, θ_0 = angle of incidence
$\vec{k}_1$ is reflected wave, θ_1 = angle of reflection
$\vec{k}_2$ is the transmitted or "refracted" wave, θ_2 = angle of refraction

let each wave be given by

$$\vec{F}_n(\vec{r},t) = \vec{F}_n e^{i(\vec{k}_n\cdot\vec{r} - \omega_n t)}$$

where $\vec{F}_n$ can be either $\vec{E}_n$ or $\vec{H}_n$ for the electric or magnetic component of the wave

boundary condition: tangential component $\vec{E}$ must be continuous at $z=0$. If $\hat{t}$ is a vector in xy plane, and we consider $\vec{r}=0$, then

$$\Rightarrow \quad \hat{t}\cdot\vec{E}_0 e^{-i\omega_0 t} + \hat{t}\cdot\vec{E}_1 e^{-i\omega_1 t} = \hat{t}\cdot\vec{E}_2 e^{-i\omega_2 t}$$

must be true for all time. Can only happen if

$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega}$ all frequencies are equal

Now consider the same boundary condition for $\vec{\rho}$ a position vector in the xy plane at $z=0$. Since ω's all equal we can cancel out the common $e^{-i\omega t}$ factors to get

$$\hat{t} \cdot \vec{E}_0 e^{i\vec{k}_0 \cdot \vec{\rho}} + \hat{t} \cdot E_1 e^{i\vec{k}_1 \cdot \vec{\rho}} = \hat{t} \cdot \vec{E}_2 e^{i\vec{k}_2 \cdot \vec{\rho}}$$

this must be true for all $\vec{\rho}$. Can only happen if the projections of the $\vec{k}_n$ in the xy plane are all equal

$$\boxed{\begin{aligned} k_{0x} &= k_{1x} = k_{2x} \\ k_{0y} &= k_{1y} = k_{2y} \end{aligned}}$$

only z components $\vec{k}$ vectors can be different

Choose coord system as in diagram so that all $\vec{k}$ vectors lie in the xz plane (y is out of page)

Since ϵ_a is real and positive, $\vec{k}_2$ and $\vec{k}_1$ are real vectors

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_1| \sin\theta_1$$

since $k_0^2 = \frac{\omega^2}{c^2} \mu_a \epsilon_a$ and $k_1^2 = \frac{\omega^2}{c^2} \mu_a \epsilon_a$

then $|\vec{k}_0| = |\vec{k}_1|$ so $\sin\theta_0 = \sin\theta_1$

$$\boxed{\theta_0 = \theta_1}$$

angle of incidence = angle of reflection

If ϵ_b is also real and positive (B is transparent)
then $|\vec{k}_2|$ is real

$$k_{0x} = k_{2x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_2| \sin\theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_b$$

$$\Rightarrow \sqrt{\mu_a \epsilon_a} \sin\theta_0 = \sqrt{\mu_b \epsilon_b} \sin\theta_2$$

in terms of index of refraction $n = \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\epsilon}}{c}\frac{c}{\omega}$

$$n = \sqrt{\mu\epsilon}$$

$$\Rightarrow n_a \sin\theta_0 = n_b \sin\theta_2$$

$$\boxed{\frac{\sin\theta_2}{\sin\theta_0} = \frac{n_a}{n_b}}$$

Snell's Law
true for all types of waves, not just EM waves

If $n_a > n_b$ then $\theta_2 > \theta_0$
In this case, when θ_0 is too large, we will have
$\frac{n_a}{n_b} \sin\theta_0 > 1$ and there will be no solution for θ_2
$\Rightarrow$ no transmitted wave

This is "total internal reflection" – wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

$$\frac{n_a}{n_b} \sin\theta_c = 1 \quad , \quad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)}$$