

For arbitrary charge distributions - not pure harmonic

For $\vec{p}_\omega e^{-i\omega t}$ pure harmonic oscillation, we found the radiated fields in electric dipole approx are

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}, \quad \vec{B} = \vec{B}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega = -k^2 \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega) = -\frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$\vec{B}_\omega = k^2 \frac{e^{ikr}}{r} (\hat{r} \times \vec{p}_\omega) = \frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} (\hat{r} \times \vec{p}_\omega)$$

$$\text{as } k = \frac{\omega}{c}$$

For an arbitrarily time varying charge distribution with electric dipole moment

$$\vec{p}(t) = \int \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t}$$

then solution for fields given by superposition

$$\vec{E}(\vec{r}, t) = \int \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$= - \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2} \right) \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$= -\frac{1}{c^2 r} \hat{r} \times \left[\hat{r} \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \omega^2 \right]$$

$$= \frac{1}{c^2 r} \hat{r} \times \left[\hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \right]$$

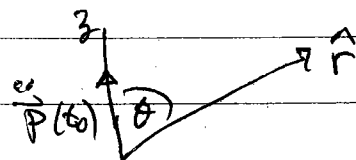
$$\boxed{\vec{E}(\vec{r}, t) = \frac{1}{c^2 r} \hat{r} \times \left[\hat{r} \times \ddot{\vec{p}}(t - r/c) \right]}$$

$$\ddot{\vec{p}} = \frac{d^2 \vec{p}}{dt^2}$$

define $t_0 \equiv t - r/c$ = "retarded time"

in spherical coords, if $\ddot{\vec{p}}(t_0)$ is along $\hat{z}$

$$\vec{E}(\vec{r}, t) = \frac{\ddot{p}(t_0)}{c^2 r} \sin \theta \hat{\theta}$$



Similarly

$$\vec{B}(\vec{r}, t) = \int \frac{d\omega}{2\pi} \vec{B}_\omega e^{-i\omega t}$$

$$= \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2} \right) (\hat{r} \times \vec{p}_\omega)$$

$$= \frac{-1}{c^2 r} \hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega$$

$$\boxed{\vec{B}(\vec{r}, t) = -\frac{1}{c^2 r} \hat{r} \times \ddot{\vec{p}}(t_0)}$$

$$\vec{B}(\vec{r}, t) = \frac{\ddot{p}(t_0)}{c^2 r} \sin \theta \hat{\phi} \quad \text{in spherical coords}$$

Poynting vector

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} \left(\frac{1}{c^2 r} \right)^2 [\ddot{p}(t_0)]^2 \sin^2 \theta \hat{r}$$

Total power radiated through a sphere of radius r is

$$\begin{aligned}
 P &= \oint da \hat{r} \cdot \vec{S} = 2\pi \int_0^\pi d\theta \sin\theta r^2 \hat{r} \cdot \vec{S} \\
 &= \frac{[\ddot{\vec{p}}(t_0)]^2}{2c^3} \underbrace{\int_0^\pi d\theta \sin^3\theta}_{4/3}
 \end{aligned}$$

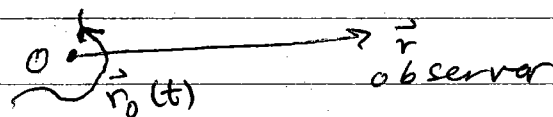
$$P = \frac{2 [\ddot{\vec{p}}(t_0)]^2}{3c^3}$$

For a point charge moving along a trajectory $\vec{r}_0(t)$

$$\vec{p}(t) = q \vec{r}_0(t)$$

$$\ddot{\vec{p}}(t) = q \ddot{\vec{r}}_0(t) = q \vec{a}(t)$$

↑ acceleration



$$P = \frac{2}{3} \frac{q^2 a^2(t_0)}{c^3}$$

Larmor's formula

← total power passing through a sphere of radius r at time t is due to acceleration at retarded time $t_0 = t - r/c$

power radiated $\propto (\text{acceleration})^2$

Larmor's formula above only holds in the non-relativistic limit since it is based on the electric dipole approx.

To go beyond non-relativistic limit, one can do the following:

- 1) transform to a new inertial frame of reference in which the charge is instantaneously at rest
- 2) apply non-relativistic Larmor formula in this frame
- 3) transform back to "lab" frame

Back to non-relativistic case

- radiation fields from moving point charge

$$\vec{E}(\vec{r}, t) = \frac{q}{c^2 r} \hat{r} \times (\hat{r} \times \vec{a}(t_0))$$

$$\vec{B}(\vec{r}, t) = -\frac{q}{c^2 r} (\hat{r} \times \vec{a}(t_0))$$

Poynting vector

$$\begin{aligned} \vec{S} &= \frac{c}{4\pi} \vec{E} \times \vec{B} = -\frac{q}{4\pi} \left(\frac{q}{c^2 r}\right)^2 [\hat{r} \times (\hat{r} \times \vec{a})] \times [\hat{r} \times \vec{a}] \\ &= \frac{c}{4\pi} \left(\frac{q}{c^2 r}\right)^2 \left\{ \hat{r} (\hat{r} \times \vec{a})^2 - (\hat{r} \times \vec{a}) [\hat{r} \cdot (\hat{r} \times \vec{a})] \right\} \\ &= \frac{q^2}{4\pi c^3} \frac{[\hat{r} \times \vec{a}(t_0)]^2}{r^2} \hat{r} \end{aligned}$$

for $\vec{a} = a \hat{z}$

$$\begin{aligned} \vec{S} &= \frac{q^2}{4\pi c^3 r^2} (a^2 - (\hat{r} \cdot \vec{a})^2) \hat{r} \\ &= \frac{q^2}{4\pi c^3} \frac{a^2(t_0)}{r^2} \sin^2 \theta \hat{r} \end{aligned}$$

$\vec{a}$ evaluated at t_0