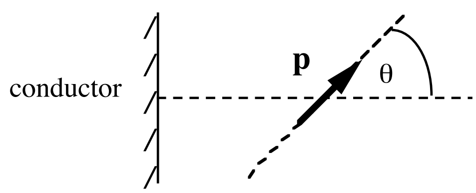
PHY 415 **Final Exam** **Fall 2007**

1) a) [10 points] An ideal point dipole $\mathbf{p}$ is positioned a distance d in front of an infinite flat grounded ($\phi = 0$) conducting plane, and oriented as shown in the figure below. Explain how you would compute the surface charge which is induced on the surface of the conductor. You do not have to do the calculation, but you must give sufficient detail to set the solution up.



b) [10 points] Two circular wire loops, each of radius R, are centered about the origin and lying in the xy plane. One loop is at height $z = +a$ and has a current I circulating clockwise. The other loop is at height $z = -a$ and has a current I circulating counter-clockwise. For large distances $r \gg R, r \gg a$ the magnetic field will decay, to leading order, proportional to $1/r^n$. What do you think will be the value of the integer n? Explain your reasoning.

2) [20 points] Consider a dielectric sphere of radius R with dielectric constant ϵ. At the center of the sphere is a point charge q. The sphere is placed in a uniform external electric field $\mathbf{E}_0 = E_0\hat{z}$.

a) What is the electrostatic potential inside and outside the sphere?

b) What is the total surface charge density induced on the surface of the sphere?

3) [30 points] For homework, you considered the effect of a static uniform magnetic field on the propagation of electromagnetic waves in a dielectric (the Faraday effect). In this problem, you will consider the effect of a static uniform magnetic field on the propagation of electromagnetic waves in a conductor. Assume we have a conductor with n conduction electrons per unit volume, and the effect of the bound electrons can be ignored (i.e. $\epsilon_b = 1$). Suppose there is a constant and uniform magnetic field along the z axis, $\mathbf{B}_0 = B_0\hat{z}$.

a) Consider now an oscillating electric field $\mathrm{Re}[\mathbf{E}_\omega \mathrm{e}^{-i\omega t}]$ with $\mathbf{E}_\omega$ perpendicular to $\mathbf{B}_0$. For a circularly polarized electric field, i.e. $\mathbf{E}_\omega = E_0(\hat{x} \pm i\hat{y})$, show that the current density of the conduction electrons is $\mathrm{Re}[\mathbf{j}_\omega \mathrm{e}^{-i\omega t}]$ where $\mathbf{j}_\omega = j_0(\hat{x} \pm i\hat{y})$, and

$$j_0 = \frac{(ne^2\tau/m)}{1 - i(\omega \mp \omega_c)\tau} E_0 \tag{1}$$

where $-e$ and m are the charge and mass of the conduction electrons, τ is the relaxation time of the electrons, and $\omega_c \equiv eB_0/mc$ is the cyclotron frequency. (*Hint*: treat the electrons as classical particles moving in the Lorentz force of $\mathbf{E}$ and $\mathbf{B}_0$ with a damping force $-m\mathbf{v}/\tau$.)

b) Consider now a circularly polarized electromagnetic wave propagating in the z direction through the conductor, with electric field $\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}[E_0(\hat{x} \pm i\hat{y})\mathrm{e}^{i(kz-\omega t)}]$. Show that k and ω are related by the dispersion relation $k^2 = (\omega/c)^2\epsilon(\omega)$, with

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega}\left(\frac{1}{\omega \mp \omega_c + i/\tau}\right) \quad , \tag{2}$$

where we assumed $\mu = 1$ and $\omega_p \equiv \sqrt{4\pi ne^2/m}$ is the plasma frequency.

c) For very low frequency waves such that $\omega \ll \omega_c$ and large magnetic fields such that $1/\tau \ll \omega_c$, show that $\omega = \omega_c(k^2c^2/\omega_p^2)$, i.e. $\omega \sim k^2$. Such waves are known as "helicons".

4) [30 points] A charge q is moving in the xy plane at $z = 0$ in an elliptical orbit centered on the origin, with constant angular velocity. The trajectory $\mathbf{r}_0(t)$ is given by, $\mathbf{r}_0(t) = a\cos(\omega t)\hat{x} + b\sin(\omega t)\hat{y}$. Assume $b < a$. Compute the time averaged Poynting vector $\langle \mathbf{S}(\mathbf{r}) \rangle$ in spherical coordinates with respect to the above x, y, z axes. Compute the radiated power per unit solid angle $dP/d\Omega$ at spherical angles (θ, φ). Make a polar plot of $dP/d\Omega$ vs. θ for the specific cases of $\varphi = 0$ and $\varphi = \pi/2$. You may use the electric dipole approximation in the radiation zone limit.

Hint: For a charge distribution with oscillating dipole moment $\mathbf{p}(t) = \mathrm{Re}[\mathbf{p}_\omega \mathrm{e}^{-i\omega t}]$, the electric dipole approximation for the electric and magnetic fields in the radiation zone gives,

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \mathrm{Re}\left[-k^2 \frac{\mathrm{e}^{i(kr-\omega t)}}{r} \hat{r} \times (\hat{r} \times \mathbf{p}_\omega)\right] \\ \mathbf{B}(\mathbf{r}, t) &= \mathrm{Re}\left[k^2 \frac{\mathrm{e}^{i(kr-\omega t)}}{r} \hat{r} \times \mathbf{p}_\omega\right] \end{aligned}$$

where $k = \omega/c$.