Magnetostatics

Lorentz Force

a charge q, in motion with velocity $\vec{v}$, feels the force

$$\vec{F} = q(\vec{E} + k_4\, \vec{v}\times\vec{B}) \quad \leftarrow \text{Lorentz force}$$

$\vec{B}$ is the magnetic field at the position of the charge. k_4 is a universal constant.

Just as the constant k_1 fixed the units of charge q, the constant k_4 can be viewed as fixing the units of B magnetic field. By choosing the units of q and B appropriately, we are free to choose any values for k_1 and k_4.

Magnetic field $\vec{B}$ is generated by moving charge. A charge q' with velocity $\vec{v}'$ ($v' \ll c$) located at the origin $\vec{r}'=0$ produces a magnetic field at position $\vec{r}$,

holds only nonrelativistically $\longrightarrow$

$$\vec{B}(\vec{r}) = k_5\, q' \frac{\vec{v}'\times\vec{r}}{r^3} = \frac{k_5}{k_1}\, \vec{v}'\times\vec{E}(\vec{r})$$

k_5 is a universal constant. we will see that it cannot be chosen independently of k_1 and k_4. (since k_1 fixed units of q, and k_4 fixed units of $\vec{B}$, there are no further new quantities whose units could be adjusted to allow us to fix k_5 arbitrarily)

The force on a charge q at position $\vec{r}$, moving with velocity $\vec{v}$, due to a charge q' at the origin moving with velocity $\vec{v}'$ is, in non-relativistic limit $(v, v' \ll c)$,

$$\vec{F} = k_1 q q' \frac{\vec{r}}{r^3} + k_4 k_5 q q' \vec{v} \times \frac{(\vec{v}' \times \vec{r})}{r^3}$$

Coulomb force — magnetic analog of Coulomb force

The magnetic part is just the point charge equivalent of the Biot-Savart law for the force between current ~~carry~~ carrying wires. If we regard $q\vec{v} = \vec{I}$ as the current of charge q, and $q'\vec{v}' = \vec{I}'$ as the current of charge q', then the magnetic force is $k_4 k_5 \vec{I} \times (\vec{I}' \times \frac{\vec{r}}{r^3})$ which is the Biot-Savart Law.

Rewrite above force as

$$\vec{F} = k_1 \left(1 + \frac{k_4 k_5}{k_1} \vec{v} \times \vec{v}' \times \right) \frac{\vec{r}}{r^3} q q'$$

we see that $\left(\frac{k_4 k_5}{k_1}\right)$ has units of $(\text{velocity})^{-2}$

it must be independent of whatever convention one used to choose the units of q or B (ie independent of choices for k_1 and k_4). Experimentally it is found that

$$\left(\frac{k_4 k_5}{k_1}\right) = \frac{1}{c^2}$$

c = speed of light in vacuum

Continuum current density

For charges q_i at positions $\vec{r}_i(t)$ with $\vec{v}_i = \frac{d\vec{r}_i}{dt}$
we define the current density

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i(t)\, \delta(\vec{r}-\vec{r}_i(t))$$

units of $\vec{j}$ are $(\text{charge})\left(\frac{\text{length}}{\text{time}}\right)\left(\frac{1}{\text{length}^3}\right) = \left(\frac{\text{charge}}{\text{area}\cdot\text{time}}\right)$

charge per unit area per unit time

For a surface S

$$\int_S da\, \hat{n}\cdot\vec{j} = I$$ current (charge per unit time) passing through surface S

Charge Conservation — vol V bounded by surface S

$$\frac{d}{dt}\int_V d^3r\, \rho(\vec{r},t) = -\oint_S da\, \hat{n}\cdot\vec{j}$$

rate of change of total charge in V = $(-)$ charge flowing out of V through S per unit time

Use $\oint_S da\,\hat{n}\cdot\vec{j} = \int_V d^3r\, \vec{\nabla}\cdot\vec{j} = -\int_V d^3r\, \frac{\partial}{\partial t}\rho(\vec{r},t)$

$\Rightarrow$ local charge conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$

A static situation has $\frac{\partial \rho}{\partial t} = 0$

$\Rightarrow$ magnetostatics is defined by the condition $\underline{\vec{\nabla} \cdot \vec{j} = 0}$

Differential formulation of Biot-Savart

For a set of charges q_i at $\vec{r}_i$ we have

$$\vec{B}(\vec{r}) = \sum_i k_5 \, q_i \, \vec{v}_i \times \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

$$= k_5 \int d^3r' \, \vec{j}(\vec{r}\,') \times \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}$$

$$= k_5 \int d^3r' \, \vec{j}(\vec{r}\,') \times \vec{\nabla}\left(\frac{-1}{|\vec{r} - \vec{r}\,'|}\right)$$

$$\vec{B}(\vec{r}) = k_5 \, \vec{\nabla} \times \left[\int d^3r' \, \frac{\vec{j}(r')}{|\vec{r} - \vec{r}\,'|}\right]$$

where we used $\vec{\nabla} \times (\vec{A}\varphi) = -\vec{A} \times \vec{\nabla}\varphi$ when $\vec{A}$ is indep of $\vec{r}$

$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{B} = 0}$ since $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ for any vector function $\vec{A}$

integral form $\oint_S da \, \hat{n} \cdot \vec{B} = 0$

$$\vec{\nabla} \times \vec{B} = k_5 \, \vec{\nabla} \times \left[\vec{\nabla} \times \left(\int d^3r' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\right)\right]$$

use $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\vec{\nabla}\times\vec{B} = k_5 \vec{\nabla}\left[\int d^3r' \, \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\right]$$

$$- k_5 \int d^3r' \, \vec{J}(\vec{r}')\, \nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

in the 2nd term, $\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\delta(\vec{r}-\vec{r}')$

in the 1st term, $\vec{\nabla}\cdot\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \vec{J}(\vec{r}')\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

$$= -\vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

since $\vec{\nabla} = -\vec{\nabla}'$

So $\int d^3r' \, \vec{\nabla}\cdot\left(\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) = -\int d^3r' \, \vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

integrate by parts $\qquad = \int d^3r' \left(\vec{\nabla}'\cdot\vec{J}(r')\right)\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

surface term $\to 0$ as
we take surface $\to \infty$
since $\vec{J}\to 0$ as $r\to\infty$

But for magnetostatics $\vec{\nabla}\cdot\vec{J} = 0 \Rightarrow$ only 2nd term remains

Thus, for magnetostatics

$$\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} \qquad \text{Ampere's law}$$

integral form $\oint_C d\vec{l}\cdot\vec{B} = 4\pi k_5 \int_S da\, \hat{n}\cdot\vec{J}$

curve bounding surface

Although above diff eqn's were derived starting from a "non-relativistic" point-charge Biot-Savart law, the actually remain true for all magnetostatic situations

So far electrostatics $\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi k_1 \rho & \text{Gauss} \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$

magnetostatics $\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} & \text{Ampere} \end{cases}$

current conservation $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

Time Dependent situations

Faraday's law of induction $\vec{\nabla}\times\vec{E} \neq 0$!

$$\oint_C d\vec{l}\cdot\vec{E} = -k_3 \frac{\partial}{\partial t} \int_S da\, \hat{n}\cdot\vec{B} \qquad \mathcal{E}_{mf} = -\frac{d\Phi}{dt}$$

emf around loop

~~mag~~ flux

voltage around closed loop ~ −time rate of change of magnetic flux through loop

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -k_3 \frac{\partial \vec{B}}{\partial t}} \qquad k_3 \text{ is universal constant}$$

Maxwell correction to Ampere's law

in our derivation of $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J}$ we used $\vec{\nabla}\cdot\vec{J} = 0$. This is only true for magnetostatics – it is NOT true in general

Alternatively, since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ always, if Amperes law was true, we would necessarily conclude that $\vec{\nabla}\cdot\vec{J} = 0$. But $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$ in general.

Proposed correction: $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \vec{W}$
where $\vec{W}$ must be such that charge conservation holds.

Now

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = 4\pi k_5 \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{W}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{W} = -4\pi k_5 \vec{\nabla}\cdot\vec{J} = 4\pi k_5 \frac{\partial \rho}{\partial t}$$ by charge conser

$$= \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E}$$ by Gauss Law

$$\Rightarrow \vec{W} = \frac{k_5}{k_1}\frac{\partial \vec{E}}{\partial t}$$

~~So~~ corrected Amperes law is

$$\boxed{\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \frac{k_5}{k_1}\frac{\partial \vec{E}}{\partial t}}$$

$\Rightarrow$ EM waves

Now consider $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$
$= -\nabla^2\vec{B}$ as $\vec{\nabla}\cdot\vec{B} = 0$

If there are no sources ($\rho = 0$, $\vec{J} = 0$) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} = \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}$$

$$= -\frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2}$$ by Faraday

$$\nabla^2 B = \frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2}$$ this is the wave equation

$\Rightarrow \frac{k_5 k_3}{k_1}$ has units of $(\text{velocity})^{-2}$

Since we know that the above wave equation describes electromagnetic waves, ie light, then

$$\frac{k_5 k_3}{k_1} = \frac{1}{c^2}$$

we already had $\quad \frac{k_4 k_5}{k_1} = \frac{1}{c^2}$

$$\Rightarrow k_3 = k_4$$

$\Rightarrow$ k_1 and k_4 are arbitrary - they can be chosen to be anything by adjusting the units of q and B. k_3 and k_5 are then fixed by $\frac{k_4 k_5}{k_1} = \frac{1}{c^2}$ $\cancel{\frac{k_5 k_3}{k_1}}$, $k_3 = k_4$

<u>Popular systems of E + M units</u>

	k_1	$k_3 = k_4$	k_5	
MKS or SI	$\frac{1}{4\pi\varepsilon_0}$	1	$\frac{\mu_0}{4\pi}$	$\left(\varepsilon_0\mu_0 = \frac{1}{c^2}\right)$
Gaussian or CGS	1	$\frac{1}{c}$	$\frac{1}{c}$	
Rationalized Gaussian	$\frac{1}{4\pi}$	$\frac{1}{c}$	$\frac{1}{4\pi c}$	

In MKS, charges are measured in "coulombs"
current measured in "amps"
magnetic field measured in "tesla" = "weber/m^2"