The Coulomb problem as a boundary value problem

Consider a conducting sphere of radius R with net charge q (as $R \to 0$ we get a point charge). What is $\phi(\vec{r})$? What is $E(\vec{r})$?

Review: Properties of conductors in electrostatics

1) $\vec{E}=0$ inside conductor – if $\vec{E}\neq 0$ then a current $\vec{J}=\sigma\vec{E}$ flows and it is not statics (σ is conductivity)
2) $\rho=0$ inside conductor – if $\vec{E}=0$ inside, then $\vec{\nabla}\cdot\vec{E}=4\pi\rho=0$
3) Any net charge on the conductor must lie on the surface – follows from (2)
4) $\phi=$ constant throughout conductor – if $\vec{E}=0$ then $\vec{E}=-\vec{\nabla}\phi \Rightarrow \phi$ is constant
5) Just outside the conductor, $\vec{E}$ is $\perp$ to surface.
   - If $\vec{E}$ has a component $\parallel$ to surface then it exerts a force on electrons at the surface leading to a surface current – so would not be static

For conducting sphere, $\rho=0$ for $r>R$ and $r<R$ all charge is on the surface $\Rightarrow \nabla^2\phi=0$ for $\begin{cases} r>R \\ r<R \end{cases}$

spherical symmetry $\Rightarrow$ expect spherically symmetric solution

$\Rightarrow \phi(\vec{r})$ depends only on $r=|\vec{r}|$

⇒ Solve Laplace's eqn by writing ∇^2 in spherical coords. Only the radial terms do not vanish.

$$\nabla^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 0$$

$$r^2\frac{d\phi}{dr} = -C_0 \quad \text{a constant}$$

$$\frac{d\phi}{dr} = -\frac{C_0}{r^2}$$

$$\phi(r) = \frac{C_0}{r} + C_1 \,, \quad C_1 \text{ a constant}$$

"outside" $r > R$ $\quad \phi^{out}(r) = \frac{C_0^{out}}{r} + C_1^{out}$

"inside" $r < R$ $\quad \phi^{in}(r) = \frac{C_0^{in}}{r} + C_1^{in}$

solution "outside" does not necessarily go smoothly into the solution "inside" because of the charge layer at $r = R$ that seperates the two regions. We need to determine the constants C_0^{in}, C_0^{out}, C_1^{in}, C_1^{out} by applying <u>boundary conditions</u> corresponding to the physical situation.

① For $r > R$, assume $\phi \to 0$ as $r \to \infty$ - boundary condition at infinity

$$\Rightarrow C_1^{out} = 0$$

$$\phi^{out}(r) = \frac{C_0^{out}}{r}$$ recover the expected Coulomb form.

2) For $r < R$.

i) We could use the fact that the region $r<R$ is a conductor with $\phi = \text{constant}$ to conclude $C_0^{in} = 0$

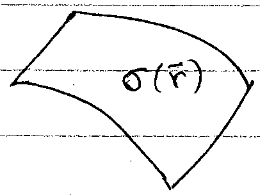
ii) or, if we were dealing with a charged shell instead of a conductor, we could argue as follows:

no charge at origin $r=0$ $\Rightarrow$ expect ϕ should be <u>finite</u> at origin $\Rightarrow$ $C_0^{in} = 0$

So $\quad \Phi^{in}(r) = C^{in} \quad$ a constant

3) Now we need boundary condition at $r=R$ where "inside" and "outside" meet.

Review: <u>Electric field and potential at a surface charge layer</u>

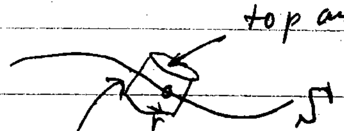


← a general surface S with surface charge density $\sigma(\vec{r})$ for $\vec{r}$ on S. $\sigma(\vec{r})da$ is total charge in area da on surface

i) Take "Gaussian pillbox" surface about point $\vec{r}$ on the surface S

top and bottom areas of pill box da

side view



side of pillbox $d\ell$

Gauss' Law in integral form $\displaystyle\oint_S da\, \hat{n}\cdot\vec{E} = 4\pi Q_{\text{enclosed}}$

expect $\vec{E}$ is finite $\Rightarrow$ contribution from sides of pillbox vanish as $dl \to 0$.

$$\oint_S da\, \hat{n}\cdot\vec{E} = \int_{top} da\, \hat{n}\cdot\vec{E} + \int_{bottom} da\, \hat{n}\cdot\vec{E}$$

$$= \left(\hat{n}^{top}\cdot\vec{E}^{top} + \hat{n}^{bottom}\cdot\vec{E}^{bottom}\right) da$$ since da is small

$\vec{E}^{top}$ is electric field at $\vec{r}$ just above the surface S

$\vec{E}^{bottom}$ is electric field at $\vec{r}$ just below the surface S

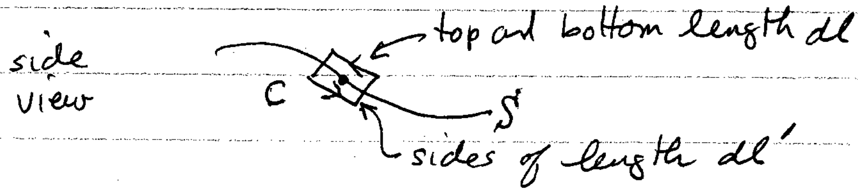
$\hat{n}^{top} \equiv \hat{n}$ is outward normal on top.

$\hat{n}^{bottom} = -\hat{n}$ is outward normal on bottom

$$\Rightarrow \left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot\hat{n}\, da = 4\pi Q_{enclosed} = 4\pi\sigma(\vec{r})\, da$$

$$\boxed{\left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot\hat{n} = 4\pi\sigma(\vec{r})}$$ <u>discontinuity</u> in <u>normal</u> <u>component</u> of $\vec{E}$

ii) Take "Amperian loop" C at surface about point $\vec{r}$.



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \oint_C d\vec{l}\cdot\vec{E}$ since $\vec{E}$ is finite at surface, if take sides $dl' \to 0$ their contribution to integral vanishes

$$\Rightarrow \oint_C d\vec{l}\cdot\vec{E} = \boxed{\left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot d\vec{l} = 0}$$

where $d\vec{l}$ is any infinitesimal tangent to the surface at $\vec{r}$.

$\Rightarrow$ tangential component of $\vec{E}$ is continuous

combine above to write $\boxed{\vec{E}^{\,top} - \vec{E}^{\,bottom} = 4\pi\sigma(\vec{r})\,\hat{n}}$

iii) $\vec{E} = -\vec{\nabla}\phi \Rightarrow \phi(r_2) - \phi(r_1) = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}\cdot\vec{E}$

Take $\vec{r}_2$ just above $\vec{r}$ on surface
$\vec{r}_1$ just below $\vec{r}$ on surface $\Big\}\ d\vec{l} \to 0$

Since $\vec{E}$ is finite $\Rightarrow \int d\vec{l}\cdot\vec{E} \to 0$

$\Rightarrow \boxed{\phi^{top} = \phi^{bottom}}$

potential ϕ is continuous at surface charge layer

can rewrite (i) as

$$(-\vec{\nabla}\phi^{top} + \vec{\nabla}\phi^{bottom})\cdot\hat{n} = 4\pi\sigma$$

$$\boxed{-\frac{\partial\phi^{top}}{\partial n} + \frac{\partial\phi^{bottom}}{\partial n} = 4\pi\sigma}$$

directional derivative of ϕ in direction $\hat{n}$

discontinuity in normal derivative of ϕ at surface

Apply to conducting sphere

ϕ continuous $\Rightarrow \phi^{in}(R) = \phi^{out}(R)$

$$C_1^{in} = \frac{C_0^{out}}{R}$$

only one unknown left

normal derivative of ϕ is discontinuous

$$-\frac{\partial \phi^{top}}{\partial n} + \frac{\partial \phi^{bottom}}{\partial n} = 4\pi\sigma$$

here $\hat{n} = \hat{r}$ the radial direction

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr}\right]_{r=R} = 4\pi\sigma$$

but $\frac{d\phi^{in}}{dr} = 0$ as $\phi^{in} = \text{constant}$

$$-\frac{d\phi^{out}}{dr}\bigg|_{r=R} = 4\pi\sigma$$

charge q is uniformly distributed on surface at R

$$-\frac{d}{dr}\left(\frac{C_0^{out}}{r}\right)_{r=R} = \frac{C_0^{out}}{R^2} = 4\pi\sigma = 4\pi\left(\frac{q}{4\pi R^2}\right) = \frac{q}{R^2}$$

$$\Rightarrow C_0^{out} = q \quad , \quad C^{in} = \frac{C^{out}}{R} = \frac{q}{R}$$

$$\phi(r) = \begin{cases} \frac{q}{R} & r < R \quad \text{inside} \\ \frac{q}{r} & r > R \quad \text{outside} \end{cases}$$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi = -\frac{d\phi}{dr} = \begin{cases} 0 & r<R \quad \text{inside} \\ \frac{q}{r^2} & r>R \quad \text{outside} \end{cases}$$

we get familiar Coulomb solution!

Summary We can view the preceeding solution for ϕ^{out} as solving Laplace's eqn $\nabla^2\phi = 0$ subject to a specified boundary condition on the normal derivative of ϕ at the boundary $r=R$ of the "outside" region of the system.

Alternate problem:

Another physical situation would be to connect conducting sphere to a battery that charges the sphere to a fixed voltage ϕ_0 (stat volts!) with respect to ground $\phi = 0$ at $r \to \infty$.

As before, outside the sphere $\phi = \frac{c_0}{r}$
Now the boundary condition is to specify the value of ϕ on the boundary of the outside region, ie

$$\phi(R) = \phi_0$$

$$\Rightarrow \quad \frac{c_0}{R} = \phi_0 \, , \quad c_0 = \phi_0 R$$

$$\phi(r) = \phi_0 \frac{R}{r}$$

(from preceeding solution, we know that charging the sphere to voltage ϕ_0 (statvolts) induces a net charge $q = \phi_0 R$ on it)

These two versions of the conducting sphere problem are examples of a more general <u>boundary value problem</u>

Solve $\nabla^2\phi = 0$ in a given region of space subject to one of the following two types of boundary conditions on the bounding surfaces of the region

i) Neumann boundary condition

$\frac{\partial \phi}{\partial n}$ - normal derivative of ϕ is specified on the bounding surface

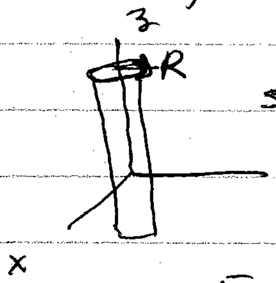
ii) Dirichlet boundary condition

ϕ - value of ϕ is specified on the bounding surfaces

If the bounding surfaces consist of <u>disjoint</u> pieces, it is possible to specify either (i) or (ii) on each piece separately to get a mixed boundary value problem.

Some more problems

infinite conducting wire of radius R with line charge density λ = charge per unit length



surface charge $\sigma = \frac{\lambda}{2\pi R}$

Expect cylindrical symmetry $\Rightarrow$ ϕ depends only on cylindrical coord r.

$\nabla^2\phi = 0$ for $r > R$, $r < R$

use ∇^2 in cylindrical coords – only radial term non vanishing

$$\nabla^2\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = 0$$

$$r\frac{d\phi}{dr} = C_0 \quad \text{constant}$$

$$\frac{d\phi}{dr} = \frac{C_0}{r}$$

$$\phi(r) = C_0 \ln r + C_1 \quad \text{const}$$

note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

one needs to fix zero of ϕ at some other radius. a convenient choice is $r = R$, but any other choice could also be made.

$$\phi^{out} = C_0^{out} \ln r + C_1^{out}$$
$$\phi^{in} = C_0^{in} \ln r + C_1^{in}$$

ϕ^{in} = const in conductor $\Rightarrow C_0^{in} = 0$

or ϕ^{in} should not diverge as $r \to 0 \Rightarrow C_0^{in} = 0$

so $\phi^{in} = C_1^{in}$ constant

boundary condition at $r = R$

$$\left[-\frac{d\phi^{out}}{dr} + \frac{d\phi^{in}}{dr} \right]_{r=R} = 4\pi\sigma$$

$$\Rightarrow \quad -\frac{C_0^{out}}{R} = 4\pi\sigma = 4\pi\left(\frac{\lambda}{2\pi R}\right) = \frac{2\lambda}{R}$$

$$C_0^{out} = -2\lambda$$

$$\phi^{out}(r) = -2\lambda \ln r + C_1^{out}$$

continuity of ϕ

$$\phi^{in}(R) = \phi^{out}(R) \quad \Rightarrow \quad C_1^{in} = -2\lambda \ln R + C_1^{out}$$

Remaining const C_1^{out} is not too important as it is just a common additive constant to both ϕ^{in} and ϕ^{out} $\Rightarrow$ does not change $\vec{E} = -\vec{\nabla}\phi$.

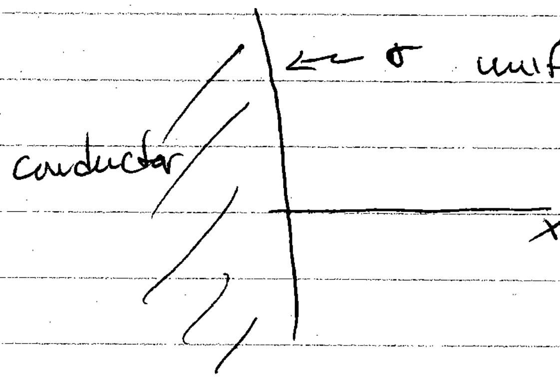
If use the condition $\phi(R) = 0$ then we can solve for C_1^{out}.

$$0 = -2\lambda \ln R + c_1^{out} \quad \Rightarrow \quad c_1^{out} = 2\lambda \ln R$$

$$\Rightarrow \quad \phi(r) = \begin{cases} -2\lambda \ln(r/R) & r > R \\ 0 & r < R \end{cases}$$

$$\Rightarrow \quad \vec{E}(\vec{r}) = \begin{cases} \frac{2\lambda}{r}\,\hat{r} & r > R \\ 0 & r < R \end{cases}$$

infinite conducting half space



← σ uniform surface charge density

expect ϕ depends only on x

$$\nabla^2 \phi = \frac{d^2\phi}{dx^2} = 0$$

$$\rightarrow \begin{cases} \phi^>(x) = c_0^> x + c_1^> & x > 0 \\ \phi^<(x) = c_0^< x + c_1^< & x < 0 \end{cases}$$

for $x < 0$, $\phi = \text{const}$ in conductor $\Rightarrow c_0^< = 0$

at $x = 0$, ϕ continuous $\Rightarrow \phi^<(0) = \phi^>(0)$

$$c_1^< = c_1^>$$

$\frac{d\phi}{dx}$ discontinuous $\Rightarrow$

$$-\left.\frac{d\phi^>}{dx}\right|_{x=0} = 4\pi\sigma$$

$$c_0^> = -4\pi\sigma$$

$$\Rightarrow \quad \phi(x) = \begin{cases} -4\pi\sigma x + c_1^> & x > 0 \\ c_1^> & x < 0 \end{cases}$$

const $c_1^>$ does not change value of $\vec{E}$

as for the wire, we cannot choose $\phi \to 0$ as $x \to \infty$.
~~we can set $\phi = 0$ at~~

$$-\vec{\nabla}\phi = \vec{E} = \begin{cases} 4\pi\sigma\,\hat{x} & x>0 \\ 0 & x<0 \end{cases}$$

<u>infinite charged plane</u>

Similar to previous problem, but now no conductor at $x<0$, just free space on both sides of the charged plane at $x=0$.

~~expect ϕ depends only on $|x|$ by symmetry~~

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = 0 \quad \Rightarrow \quad \begin{aligned} \phi^> &= c_0^> x + c_1^> & x>0 \\ \phi^< &= c_0^< x + c_1^< & x<0 \end{aligned}$$

continuity of ϕ at $x=0$

$$\rightarrow \quad \phi^>(0) = \phi^<(0) \quad \Rightarrow \quad c_1^> = c_1^<$$

discontinuity of $d\phi/dx$ at $x=0$

$$-\frac{d\phi^>}{dx} + \frac{d\phi^<}{dx} = 4\pi\sigma$$

$$-c_0^> + c_0^< = 4\pi\sigma$$

Define $\bar{c}_0 = \dfrac{c_0^> + c_0^<}{2}$

Then we can write

$$C_0^< = \bar{C}_0 + 2\pi\sigma$$
$$C_0^> = \bar{C}_0 - 2\pi\sigma$$

$$\phi = \begin{cases} -2\pi\sigma x + \bar{C}_0 x + C_1^> & x > 0 \\ 2\pi\sigma x + \bar{C}_0 x + C_1^> & x < 0 \end{cases}$$

$$\Rightarrow -\frac{d\phi}{dx} = \vec{E} = \begin{cases} (2\pi\sigma - \bar{C}_0)\,\hat{x} & x > 0 \\ (-2\pi\sigma - \bar{C}_0)\,\hat{x} & x < 0 \end{cases}$$

Const $C_1^>$ does not effect $\vec{E}$ – additive const to ϕ
$\bar{C}_0$ represents const uniform electric field $-\bar{C}_0\hat{x}$, that exists independently of the charged surface

If we assumed that all $\vec{E}$ fields are just those arising from the plane, then we can set $\bar{C}_0 = 0$. Equivalently, if the plane is the only source of $\vec{E}$, then we expect ϕ depends only on $|x|$ by symmetry. $\Rightarrow C_0^< = -C_0^>$ and again $\bar{C}_0 = 0$. In this case

$$\phi(x) = \begin{cases} -2\pi\sigma x & x > 0 \\ 2\pi\sigma x & x < 0 \end{cases}$$

(we also set $C_1^> = 0$ here corresponding to $\phi(0) = 0$

$$\vec{E}(x) = \begin{cases} 2\pi\sigma\,\hat{x} & x > 0 \\ -2\pi\sigma\,\hat{x} & x < 0 \end{cases}$$

$\vec{E}$ is constant ~~by~~ but oppositely directed on either side of the charged plane

Green's Theorem, Uniqueness, Green function part II

We want to show that the boundary value problem we described is well posed – ie there is a unique solution. We start by deriving Greens Theorems

Consider $\int_V d^3r\, \vec{\nabla}\cdot\vec{A} = \oint_S da\, \hat{n}\cdot\vec{A}$ Gauss theorem

let $\vec{A} = \phi\vec{\nabla}\psi$ ϕ, ψ any two scalar functions

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{A} = \phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$$

$$\phi\vec{\nabla}\psi\cdot\hat{n} = \phi\frac{\partial\psi}{\partial n}$$

$$\Rightarrow \int_V d^3r\left(\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\, \phi\frac{\partial\psi}{\partial n}$$ Green's 1st identity

let $\phi \leftrightarrow \psi$

$$\int_V d^3r\left(\psi\nabla^2\phi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right) = \oint_S da\, \psi\frac{\partial\phi}{\partial n}$$

subtract

$$\int_V d^3r\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \oint_S da\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)$$ Green's 2nd identity

Specifying both ϕ and $\frac{\partial \phi}{\partial n}$ on surface is known as "Cauchy" boundary conditions — for Laplace's eqn, Cauchy b.c. overspecify the problem + a solution cannot in general be found.

Uniqueness

If we have a system of charges in vol V, and either the potential ϕ, or its normal derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of V, then there is a unique solution to Poisson's equation inside V. Specifying ϕ is known as Dirichlet boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as Neumann boundary conditions.

proof: Suppose we had two solutions ϕ_1 and ϕ_2, both with $-\nabla^2\phi = 4\pi\rho$ inside V, and obeying specified b.c. on surface of V.

Define $u = \phi_2 - \phi_1 \rightarrow \nabla^2 u = 0$ inside V

and $u = 0$ on surface S - for Dirichlet b.c.
or $\frac{\partial u}{\partial n} = 0$ on surface S - for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = u$

$$\int_V d^3r\,(\underbrace{u\nabla^2 u}_{=0 \text{ as } \nabla^2 u = 0} + \vec{\nabla}u\cdot\vec{\nabla}u) = \underbrace{\oint_S da\, u\frac{\partial u}{\partial n}}_{=0 \text{ as } u \text{ or } \frac{\partial u}{\partial n} = 0}$$

$$\Rightarrow \int_V d^3r\, |\vec{\nabla} U|^2 = 0 \qquad \Rightarrow \vec{\nabla} U = 0$$
$$\Rightarrow U = \text{const}$$

For Dirichlet b.c., $U = 0$ on surface S, so const $= 0$ and $\phi_1 = \phi_2$. Solution is unique

For Neumann b.c., ϕ_1 and ϕ_2 differ only by an arbitrary constant. Since $\vec{E} = -\vec{\nabla}\phi$, the electric fields $\vec{E}_1 = -\vec{\nabla}\phi_1$ and $E_2 = -\vec{\nabla}\phi_2$ are the same.

~~Bdddd~~ If boundary ~~solution~~ surface S consists of several disjoint pieces, then solution is unique if specify ϕ on some pieces and $\frac{\partial\phi}{\partial n}$ on other pieces.

Solution of Poisson's equation with both ϕ and $\frac{\partial\phi}{\partial n}$ specified on the same surface S (Cauchy b.c.) does not in general exist, since specifying either ϕ or $\frac{\partial\phi}{\partial n}$ alone is enough to give a unique solution.