The Coulomb problem as a boundary value problem

Consider a conducting sphere of radius \( R \) with net charge \( q \) (as \( R \to 0 \) we get a point charge).

What is \( \phi(r) \)? What is \( E(r) \)?

Review: Properties of conductors in electrostatics

1) \( \vec{E} = 0 \) inside conductor - if \( \vec{E} \neq 0 \) then a current
   \( \vec{j} = \sigma \vec{E} \) flows and it is not statics (\( \sigma \) is conductivity)
2) \( \vec{j} = 0 \) inside conductor - if \( \vec{E} = 0 \) inside, then \( \nabla \cdot \vec{E} = \nabla \cdot \vec{j} = 0 \)
3) Any net charge on the conductor must lie on the surface - follows from (2)
4) \( \phi = \) constant throughout conductor - if \( \vec{E} = 0 \)
   then \( \vec{E} = -\nabla \phi \Rightarrow \phi \) is constant
5) Just outside the conductor, \( \vec{E} \) is \( \perp \) to surface.
   - If \( \vec{E} \) has a component \( \parallel \) to surface, then it exerts a force on electrons at the surface, leading to a surface current - so would not be static

For conducting sphere, \( \vec{E} = 0 \) for \( r > R \) and \( r < R \)
all charge \( \pm q \) on the surface \( \Rightarrow \nabla^2 \phi = 0 \) for \( \{r \geq R, r < R\} \)

Spherical symmetry \( \Rightarrow \) expect spherically symmetric solution

\[ \Rightarrow \phi(\hat{r}) \text{ depends only on } r = \hat{r} \hat{r} \]
Solve Laplace’s equation by writing $\nabla^2 \phi$ in spherical coordinates, only the radial terms do not vanish.

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 0$$

$$r^2 \frac{d\phi}{dr} = -C_0 \quad \text{a constant}$$

$$\frac{d\phi}{dr} = -\frac{C_0}{r^2}$$

$$\phi(r) = \frac{C_0}{r} + C_1, \quad C_1 \text{ a constant}$$

"outside" $r > R$  \hspace{1cm} \phi_{\text{out}}(r) = \frac{C_0^{\text{out}}}{r} + C_1^{\text{out}}$

"inside" $r < R$ \hspace{1cm} \phi_{\text{in}}(r) = \frac{C_0^{\text{in}}}{r} + C_1^{\text{in}}$

The solution "outside" does not necessarily go smoothly into the solution "inside" because of the charge layer at $r = R$ that separates the two regions. We need to determine the constants $C_0^{\text{in}}, C_0^{\text{out}}, C_1^{\text{in}}, C_1^{\text{out}}$ by applying boundary conditions corresponding to the physical situation.

1. For $r > R$, assume $\phi \to 0$ as $r \to \infty$ - boundary condition at infinity \hspace{1cm} $\Rightarrow C_1^{\text{out}} = 0$

$\phi(r) = \frac{C_0^{\text{out}}}{r}$ recover the expected Coulomb form.
2) For \( r < R \).

   i) We could use the fact that the region \( r < R \) is a conductor with \( \phi \) constant to conclude \( C_0 = 0 \).

   ii) Or, if we were dealing with a charged shell instead of a conductor, we could argue as follows:

   no charge at origin \( r = 0 \) \( \Rightarrow \) expect \( \phi \) should be finite at origin \( \Rightarrow C_0 = 0 \)

   So \( \phi(r) = C \) a constant

3) Now we need boundary condition at \( r = R \) where "inside" and "outside" meet.

Review: Electric field and potential at a surface charge layer

\( \sigma(r) \) \( \Rightarrow \) a general surface \( S \) with surface charge density \( \sigma(r) \) \( \Rightarrow \) for \( \mathbf{E} \) on \( S \), \( \sigma(r) \) \( \text{da} \) is total charge \( \text{in area} \) \( \text{da} \) on surface

i) Take "Gaussian pillbox" surface about point \( \mathbf{r} \) on the surface \( S \)

   top and bottom areas of pill box \( \text{da} \)

   side view

   side of pillbox \( \text{dl} \)

Gauss' Law in integral form \( \int \sigma \text{da} \cdot \mathbf{E} = 4\pi Q \text{enclosed} \)
\[ \int_S \hat{m} \cdot \vec{E} = \int_{\text{top}} \hat{m} \cdot \vec{E} + \int_{\text{bottom}} \hat{m} \cdot \vec{E} \]

\[ = (\hat{m}_{\text{top}} \cdot \vec{E}_{\text{top}} + \hat{m}_{\text{bottom}} \cdot \vec{E}_{\text{bottom}}) \, da \quad \text{since } da \text{ is small} \]

\[ \vec{E}_{\text{top}} \text{ is electric field at } \vec{r} \text{ just above the surface } S \]

\[ \vec{E}_{\text{bottom}} \text{ is electric field at } \vec{r} \text{ just below the surface } S \]

\[ \hat{m}_{\text{top}} \equiv \hat{m} \text{ is outward normal on top} \]

\[ \hat{m}_{\text{bottom}} = -\hat{m} \text{ is outward normal on bottom} \]

\[ \Rightarrow (\vec{E}_{\text{top}} - \vec{E}_{\text{bottom}}) \cdot \hat{m} \, da = 4\pi \sigma \text{ enclosed } = 4\pi \sigma (\vec{r}) \, da \]

\[ (\vec{E}_{\text{top}} - \vec{E}_{\text{bottom}}) \cdot \hat{m} = 4\pi \sigma (\vec{r}) \]

Discontinuity in normal component of \( \vec{E} \)

\[ \ii) \text{ Take "American loop" } C \text{ at surface about point } \vec{r}. \]

\[ \nabla \times \vec{E} = 0 \Rightarrow \oint_C \vec{E} \cdot dl \quad \text{since } \vec{E} \text{ is finite at surface,} \]

\[ \text{if take sides } dl \to 0 \text{ their contribution to integral vanishes} \]

\[ \Rightarrow \oint_C \vec{E} \cdot dl = (\vec{E}_{\text{top}} - \vec{E}_{\text{bottom}}) \cdot dl \to 0 \]

Where \( dl \) is any infinitesimal tangent to the surface at \( \vec{r} \).
=> tangential component of E is continuous

combine above to write

\[ \vec{E}_{\text{top}} - \vec{E}_{\text{bottom}} = 4\pi \sigma \hat{\mathbf{n}} \]

iii) \( \vec{E} = -\nabla \phi \Rightarrow \phi (r_2) - \phi (r_1) = -\int_{r_1}^{r_2} \nabla \phi \cdot d\mathbf{l} \)

Take \( r_2 \) just above \( \hat{\mathbf{n}} \) on surface \( \int \ n \Rightarrow d\mathbf{l} \rightarrow 0 \)

Since \( \vec{E} \) is finite \( \Rightarrow \int \ n \cdot \vec{E} \rightarrow 0 \)

\[ \Rightarrow \phi_{\text{top}} = \phi_{\text{bottom}} \]

potential \( \phi \) is continuous at surface charge layer

can rewrite (i) as

\[ \left( -\nabla \phi_{\text{top}} + \nabla \phi_{\text{bottom}} \right) \cdot \hat{\mathbf{n}} = 4\pi \sigma \]

\[ -\frac{\partial \phi_{\text{top}}}{\partial n} + \frac{\partial \phi_{\text{bottom}}}{\partial n} = 4\pi \sigma \]

I directional derivative of \( \phi \) in direction \( \hat{\mathbf{n}} \)

discontinuity in normal derivative of \( \phi \) at surface

Apply to conducting sphere

\( \phi \) continuous \( \Rightarrow \phi_{\text{in}} (R) = \phi_{\text{out}} (R) \)

\[ C_{\text{in}} = \frac{C_{\text{out}}}{R} \]

only one unknown left
normal derivative of \( \phi \) is discontinuous

\[- \frac{\partial \phi^{\text{top}}}{\partial n} + \frac{\partial \phi^{\text{bottom}}}{\partial n} = 4\pi \sigma\]

here \( n = \hat{r} \) the radial direction

\[
\left[ - \frac{d\phi^{\text{out}}}{dr} + \frac{d\phi^{\text{in}}}{dr} \right]_{r=R} = 4\pi \sigma
\]

but \( \frac{d\phi^{\text{in}}}{dr} = 0 \) as \( \phi^{\text{in}} \) constant

\[- \frac{d\phi^{\text{out}}}{dr} \bigg|_{r=R} = \frac{4\pi \sigma}{R} \quad \text{charge} q \text{ is uniformly distributed on surface at} \ R\]

\[- \frac{d}{dr} \left( \frac{C^{\text{out}}}{r} \right)_{r=R} = \frac{C^{\text{out}}}{R^2} = 4\pi \sigma = \frac{q}{(4\pi \epsilon_0 R^2)} = \frac{q}{R^2} \]

\[\Rightarrow C^{\text{out}} = R, \quad C^{\text{in}} = \frac{C^{\text{out}}}{R} = \frac{q}{R}\]

\[\phi(r) = \begin{cases} \frac{q}{R} & r < R \text{ inside} \\ \frac{q}{r} & r > R \text{ outside} \end{cases}\]

\[\Rightarrow \mathbf{E} = -\nabla \phi = -\frac{d\phi}{dr} = \begin{cases} 0 & r < R \text{ inside} \\ \frac{q}{r^2} & r > R \text{ outside} \end{cases}\]

we get familiar Coulomb solution!
Summary: We can view the preceding solution for \( \phi \) as solving Laplace's equation \( \nabla^2 \phi = 0 \) subject to a specified boundary condition on the normal derivative of \( \phi \) at the boundary \( r = R \) of the "outside" region of the system.

Alternate problem:
Another physical situation would be to connect a conducting sphere to a battery that charges the sphere to a fixed voltage \( \phi_0 \) (stat volts) with respect to ground \( \phi = 0 \) at \( r \to \infty \).

As before, outside the sphere \( \phi = \frac{\phi_0}{r} \)
Now the boundary condition is to specify the value of \( \phi \) on the boundary of the outside region, i.e.
\[
\phi(R) = \phi_0
\]
\[
\Rightarrow \frac{\phi_0}{R} = \phi_0, \quad C_0 = \phi_0 R
\]
\[
\phi(r) = \phi_0 \frac{R}{r}
\]
(from preceding solution, we know that charging the sphere to voltage \( \phi_0 \) (stat volts) induces a net charge \( q = \phi_0 R \) on it.)
These two versions of the conducting sphere problem are examples of a more general boundary value problem.

Solve $\nabla^2 \phi = 0$ in a given region of space subject to one of the following two types of boundary conditions on the boundary surfaces of the region:

i) Neumann boundary condition

$$\frac{\partial \phi}{\partial n} \text{ - normal derivative of } \phi \text{ is specified on the boundary surface}$$

ii) Dirichlet boundary condition

$$\phi \text{ - value of } \phi \text{ is specified on the boundary surfaces}$$

If the boundary surfaces consist of disjoint pieces, it is possible to specify either (i) or (ii) on each piece separately to get a mixed boundary value problem.
Some more problems

Infinite conducting wire of radius $R$ with line charge density $\lambda = \text{charge per unit length}$

Surface charge $\sigma = \frac{\lambda}{2\pi R}$

Expect cylindrical symmetry $\Rightarrow \phi$ depends only on cylindrical coord $r$.

$\nabla^2 \phi = 0 \quad \text{for} \quad r > R, \quad r < R$

Use $\nabla^2$ in cylindrical coords — only radial term non vanishing

$\nabla^2 \phi = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = 0$

$r \frac{d\phi}{dr} = C_0 \quad \text{constant}$

$\frac{d\phi}{dr} = \frac{C_0}{r}$

$\phi(r) = C_0 \ln r + C_1 \quad \text{const}$

Note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

One needs to fix $\phi$ at some other radius, a convenient choice is $r = R$, but any other choice could also be made.
\[
\phi_{\text{out}} = C_0 \ln r + C_1^{\text{out}} \\
\phi_{\text{in}} = C_0 \ln r + C_1^{\text{in}}
\]

\[\phi_{\text{in}} = \text{const } \text{in conductor } \Rightarrow C_0^{\text{in}} = 0\]

or \[\phi_{\text{in}} \text{ should not diverge as } r \to 0 \Rightarrow C_0^{\text{in}} = 0\]

So \[\phi_{\text{in}} = C_1^{\text{in}} \text{ constant}\]

Boundary condition at \(r = R\)

\[
\left[ -\frac{d\phi_{\text{out}}}{dr} + \frac{d\phi_{\text{in}}}{dr} \right]_{r=R} = 4\pi \sigma
\]

\[
\Rightarrow -\frac{C_0^{\text{out}}}{R} = 4\pi \sigma = 4\pi \left( \frac{1}{2\pi R} \right) = \frac{2\lambda}{R}
\]

\[C_0^{\text{out}} = -2\lambda\]

\[\phi_{\text{out}}^{\text{out}} = -2\lambda \ln R + C_1^{\text{out}}\]

Continuity of \(\phi\)

\[\phi_{\text{in}}(R) = \phi_{\text{out}}(R) \Rightarrow C_1^{\text{in}} = -2\lambda \ln R + C_1^{\text{out}}\]

Remaining const. \(C_1^{\text{out}}\) is not too important as it is just a common additive constant to both \(\phi_{\text{in}}\) and \(\phi_{\text{out}}\) \(\Rightarrow\) does not change \(\epsilon = -\nabla \phi\).

If use the condition \(\phi(R) = 0\) then we can solve for \(C_1^{\text{out}}\).
\[ \phi(r) = \begin{cases} -2A \ln(r/R) & r > R \\ 0 & r < R \end{cases} \]

\[ \mathbf{E} (r) = \begin{cases} \frac{2A}{r^2} & r > R \\ 0 & r < R \end{cases} \]

infinite conducting half space

\[ \sigma \quad \text{uniform surface charge density} \]

\[ \nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \]

\[ \begin{align*}
\phi^x(x) &= \phi^x_0 x + \phi^x_1 & x > 0 \\
\phi^x(x) &= \phi^x_0 x + \phi^x_1 & x < 0
\end{align*} \]

for \( x < 0 \), \( \phi = \text{const} \text{ at conductor} \Rightarrow \phi^x_1 = 0 \)

at \( x = 0 \), \( \phi \text{ continuous} \Rightarrow \phi^x(0) = \phi^x(0) \)

\[ \phi^x_1 = \phi^x_1 \]

\[ \frac{d \phi^x}{dx} \text{ discontinuous} \Rightarrow \]

\[ - \frac{d \phi^x}{dx} \bigg|_{x=0} = 4\pi \sigma \]

\[ c^x_0 = -4\pi \sigma \]

\[ \Rightarrow \phi(x) = \begin{cases} -4\pi \sigma x + \phi^x_1 & x > 0 \\
\phi^x_1 & x < 0 \end{cases} \]

const \( \phi^x_1 \) does not change value of \( \mathbf{E} \)
as for the wire, we cannot choose \( \phi \to 0 \) as \( x \to \infty \).

We can set \( \phi \to 0 \) not

\[
\vec{\nabla} \phi = \vec{E} = \begin{cases} 
4\pi \sigma \hat{x} & x > 0 \\
0 & x < 0 
\end{cases}
\]

infinite charged plane

Similar to previous problem, but now no conductor at \( x < 0 \), just free space on both sides of the charged plane at \( x = 0 \).

**Symmetry**

\[
\nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \quad \Rightarrow \quad \phi^+ = C_0^+ x + C_1^+ \quad x > 0 \\
\phi^- = C_0^- x + C_1^- \quad x < 0
\]

Continuity of \( \phi \) at \( x = 0 \)

\[
\phi^+(0) = \phi^-(0) \quad \Rightarrow \quad C_1^+ = C_1^-
\]

Discontinuity of \( \frac{d\phi}{dx} \) at \( x = 0 \)

\[
- \frac{d\phi^+}{dx} + \frac{d\phi^-}{dx} = 4\pi \sigma
\]

\[
- C_0^+ + C_0^- = 4\pi \sigma
\]

Define \( \bar{C}_0 = \frac{C_0^+ + C_0^-}{2} \)

Then we can write
\[ C_0^\downarrow = \overline{C_0} + 2\pi\sigma \]
\[ C_0^\uparrow = \overline{C_0} - 2\pi\sigma \]

\[
\phi = \begin{cases} 
-2\pi\sigma x + \overline{C_0} x + C_1^\downarrow & x > 0 \\
2\pi\sigma x + \overline{C_0} x + C_1^\uparrow & x < 0
\end{cases}
\]

\[
-\frac{d\phi}{dx} = \vec{E} = \begin{cases} 
(2\pi\sigma - \overline{C_0}) \hat{x} & x > 0 \\
(-2\pi\sigma - \overline{C_0}) \hat{x} & x < 0
\end{cases}
\]

Constant \( C_1 \) does not affect \( \vec{E} \) - additive constant to \( \phi \)

\( \overline{C_0} \) represents constant uniform electric field \(-\overline{C_0} \hat{x}\),

that exists independently of the charged surface.

If we assumed that all \( \vec{E} \) fields are just those arising from the plane, then we can set \( C_0 = 0 \).

Equivalently, if the plane is the only source of \( \vec{E} \),

then we expect \( \phi \) depends only on \(|x|\) by symmetry.

\[ C_1^\downarrow = -C_1^\uparrow \] and again \( C_0 = 0 \). In this case

\[
\phi(x) = \begin{cases} 
-2\pi\sigma x & x > 0 \\
2\pi\sigma x & x < 0
\end{cases}
\]

we also set \( C_1^\uparrow = 0 \) here corresponding to \( \phi(0) = 0 \)

\[
\vec{E}(x) = \begin{cases} 
2\pi\sigma \hat{x} & x > 0 \\
-2\pi\sigma \hat{x} & x < 0
\end{cases}
\]

\( \vec{E} \) is constant but oppositely directed on either side of the charged plane.
We want to show that the boundary value problem we described is well posed - i.e. there is a unique solution. We start by deriving Green's theorems.

Consider \( \int d^3r \ \nabla \cdot \mathbf{A} = \oint \text{da} \ \hat{\mathbf{n}} \cdot \mathbf{A} \) \text{ Gauss theorem}

let \( \mathbf{A} = \phi \mathbf{\hat{u}} \phi \), \( \phi, \psi \) any two scalar functions

\[ \nabla \cdot \mathbf{A} = \phi \nabla \phi + \nabla \phi \cdot \mathbf{\hat{u}} \phi \]

\[ \phi \mathbf{\hat{u}} \phi \cdot \hat{\mathbf{n}} = \phi \frac{\partial \phi}{\partial m} \]

\[ \Rightarrow \int d^3r \ (\phi \nabla^2 \psi + \nabla \phi \cdot \mathbf{\hat{u}} \phi) = \oint \text{da} \ \phi \frac{\partial \phi}{\partial m} \] \text{ Green's 1st identity}

let \( \phi \leftrightarrow \psi \)

\[ \int d^3r \ (\psi \nabla^2 \phi + \nabla \phi \cdot \mathbf{\hat{u}} \phi) = \oint \text{da} \ \psi \frac{\partial \phi}{\partial m} \]

Subtract

\[ \int d^3r \ (\phi \nabla^2 \psi - \psi \nabla^2 \phi) = \oint \text{da} \ \left( \phi \frac{\partial \phi}{\partial m} - \psi \frac{\partial \phi}{\partial m} \right) \] \text{ Green's 2nd identity}
Specifying both $\phi$ and $\frac{\partial \phi}{\partial n}$ on surface is known as
"Cauchy" boundary conditions — for Laplace's equation.
Cauchy b.c. over-specify the problem and a solution cannot in general be found.

**Uniqueness**

If we have a system of charges in vol $V$,
and either the potential $\phi$, or its normal
derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of $V$,
then there is a unique solution to Poisson's equation
inside $V$. Specifying $\phi$ is known as Dirichlet
boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as
Neumann boundary conditions.

**Proof**: Suppose we had two solutions $\phi_1$ and $\phi_2$,
both with $-\nabla^2 \phi = \rho$ inside $V$, and obeying
specified b.c. on surface of $V$.

Define $U = \phi_2 - \phi_1 \implies \nabla^2 U = 0$ inside $V$

and $U = 0$ on surface $S$ — for Dirichlet b.c.
or $\frac{\partial U}{\partial n} = 0$ on surface $S$ — for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = U$

$$\int_V \left( U \nabla^2 U + \nabla U \cdot \nabla U \right) = \int_S U \frac{\partial U}{\partial n}$$

As $\nabla^2 U = 0$ and $U \propto \frac{\partial U}{\partial n} = 0$
\[ \Rightarrow \int \nabla^2 u = 0 \Rightarrow \nabla u = 0 \Rightarrow u = \text{const} \]

For Dirichlet b.c., \( u = 0 \) on surface \( S \), so \( \text{const} = 0 \) and \( \phi_1 = \phi_2 \). Solution is unique.

For Neumann b.c., \( \phi_1 \) and \( \phi_2 \) differ only by an arbitrary constant. Since \( E = -\nabla \phi \), the electric fields \( E_1 = -\nabla \phi_1 \) and \( E_2 = -\nabla \phi_2 \) are the same.

If boundary surface \( S \) consists of several disjoint pieces, then solution is unique if specify \( \phi \) on some pieces and \( \frac{\partial \phi}{\partial n} \) on other pieces.

Solution of Poisson's equation with both \( \phi \) and \( \frac{\partial \phi}{\partial n} \) specified on the same surface \( S \) (Cauchy b.c.) does not in general exist, since specifying either \( \phi \) or \( \frac{\partial \phi}{\partial n} \) alone is enough to give a unique solution.