Separation of Variables

If the system has a rectangular boundary, we can look for solutions to $\nabla^2 \phi = 0$ of the form

$$\phi (\vec{r}) = X(x)Y(y)Z(z)$$

product of three functions, each of one variable only

$$\nabla^2 \phi = 0 \Rightarrow \frac{1}{\phi} \nabla^2 \phi = 0$$

$$\Rightarrow \frac{1}{X(x)} \frac{d^2X}{dx^2} + \frac{1}{Y(y)} \frac{d^2Y}{dy^2} + \frac{1}{Z(z)} \frac{d^2Z}{dz^2} = 0$$

The only way this can happen for all values of $x, y, z$ is if each of the three terms is a constant, call them $a^2, b^2, c^2$

$$\frac{1}{X} \frac{d^2X}{dx^2} = a^2 \Rightarrow X(x) = A_1 e^{-ax} + A_2 e^{ax}$$

$$\frac{1}{Y} \frac{d^2Y}{dy^2} = b^2 \Rightarrow Y(y) = B_1 e^{-by} + B_2 e^{by}$$

$$\frac{1}{Z} \frac{d^2Z}{dz^2} = c^2 \Rightarrow Z(z) = C_1 e^{-cz} + C_2 e^{cz}$$

with $a^2 + b^2 + c^2 = 0$

$$\Rightarrow \text{at least one of the } a^2, b^2, c^2 \text{ is } < 0$$

$$\Rightarrow \text{at least one of the } a, b, c \text{ is imaginary}$$
Above is one particular solution, but there are many solutions, each with different $a_i, b_i, c_i$, but all obeying the constraint $a_i^2 + b_i^2 + c_i^2 = 0$. The general solution is a superposition of these:

$$
\Phi(x, y, z) = \sum \left( A_{iz} e^{-a_i x} + A_{iz}^* e^{a_i x} \right) \left( B_{iz} e^{-b_i y} + B_{iz}^* e^{b_i y} \right) \left( C_i e^{-c_i z} + C_i^* e^{c_i z} \right).
$$

**Example**

$$
a_i^2 + b_i^2 + c_i^2 = 0
$$

Consider a channel shaped as below — infinite along $z$

$$
\begin{align*}
\phi(x, y) &= 0 \\
\phi(0, y) &= 0 \\
\phi(a, y) &= 0 \\
\phi(x, y) &\to 0 \text{ as } y \to \infty \\
\phi(x, 0) &= f(x) \text{ specified function}
\end{align*}
$$

Solution is independent of $z$ \Rightarrow

$$
\Phi(x, y) = \sum \left( A_{iz} e^{-a_i x} + A_{iz}^* e^{a_i x} \right) \left( B_{iz} e^{-b_i y} + B_{iz}^* e^{b_i y} \right).
$$

$$
a_i^2 + b_i^2 = 0
$$

We will see that the correct thing to choose is $a_i = i\alpha_i$ and $b_i = \alpha_i$.

$$
\Phi(x, y) = \sum \left( A_{iz} \cos \alpha_i x + B_{iz} \sin \alpha_i x \right) \left( C_i e^{-\alpha_i y} + D_i e^{\alpha_i y} \right).
$$

**Where**

$$
\begin{align*}
A_i &= A_{iz} + A_{iz}^* \\
C_i &= B_{iz} \\
B_i &= \sqrt{2} \left( A_{iz} - A_{iz}^* \right) \\
D_i &= B_{iz}
\end{align*}
$$
Now \( \phi(x, y) \to 0 \) as \( y \to \infty \) for all \( x \) \hfill [D] = 0

\[ \Rightarrow \phi(x, y) = \sum \frac{A_i}{c} \cos \alpha_i x + \frac{B_i}{c} \sin \alpha_i x \] \[ e^{-\alpha_i y} \]

where \( A_i = A_i' c \), \( B_i = B_i' c \)

\[ \phi(0, y) = 0 \Rightarrow \sum \frac{A_i}{c} e^{-\alpha_i y} = 0 \] for all \( y \) \hfill \[ A_i' = 0 \]

\[ \Rightarrow \phi(x, y) = \sum \frac{B_i}{c} \sin (\alpha_i x) e^{-\alpha_i y} \]

\[ \phi(a, y) = 0 \Rightarrow \sum \frac{B_i}{c} \sin (\alpha_i a) e^{-\alpha_i y} = 0 \] for all \( y \)

\[ \Rightarrow \sin (\alpha_i a) = 0 \quad \text{or} \quad \alpha_i a = n\pi \]

\[ \Rightarrow \phi(x, y) = \sum_{n=1}^{\infty} \frac{B_n}{c} \sin \left( \frac{n\pi x}{a} \right) e^{-\frac{n\pi y}{a}} \]

\[ \alpha_i = \frac{n\pi}{a} \quad \text{integer} \ n \geq 1 \]

Finally

\[ \phi(x, 0) = f(x) \Rightarrow \sum_{n=1}^{\infty} \frac{B_n}{c} \sin \left( \frac{n\pi x}{a} \right) = f(x) \]

This is just the Fourier series for \( f(x) \)!

\[ B_n' = \frac{2}{a} \int_0^a f(x) \sin \left( \frac{n\pi x}{a} \right) dx \]

We have determined all unknown coefficients and found the solution

See Jackson 2-8 if

Fourier series needs review
Recall orthogonality:

\[
\frac{2}{a} \int_0^a \sin \left( \frac{n \pi x}{a} \right) \sin \left( \frac{m \pi x}{a} \right) dx = \begin{cases} 
0 & m \neq n \\
1 & m = n 
\end{cases}
\]

For \( f(x) = \phi_0 \) a constant,

\[
B_n = \frac{2}{a} \phi_0 \int_0^a \sin \left( \frac{n \pi x}{a} \right) dx = \frac{2 \phi_0}{a} \left[ -\frac{a}{n \pi} \cos \left( \frac{n \pi x}{a} \right) \right]_0^a = \begin{cases} 
\frac{2 \phi_0}{n \pi} \left( 1 - \cos n \pi \right) & \text{even} \\
\frac{4 \phi_0}{n \pi} & \text{odd} 
\end{cases}
\]
Polar Coordinates - still translationally invariant along z - so two dimensional

\[ \nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

\[ \frac{r^2 \nabla^2 \phi}{\phi} = \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{\phi^2} \frac{d^2 \phi}{d\phi^2} = 0 \]

Each term must be a constant

\[ \Rightarrow \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = \nu^2, \quad \frac{1}{\phi^2} \frac{d^2 \phi}{d\phi^2} = -\nu^2 \]

Solutions are

\[ R(r) = a r^\nu + b r^{-\nu}, \quad \nu \neq 0 \]
\[ \Phi(\phi) = A \cos(\nu \phi) + B \sin(\nu \phi) \]

\[ R(r) = A_0 + B_0 \ln r, \quad \nu = 0 \]
\[ \Phi(\phi) = A_0 + B_0 \phi \]

If \( \phi \) can take its entire range from 0 to 2\( \pi \)

(such as problem in which \( \phi \) is specified on the surface of a cylinder) then periodicity in \( \phi \to \phi + 2\pi \) requires \( B_0 = 0 \) and \( \nu = \text{integer} \)

\[ \phi = A_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[ r^n (A_n \cos(n \phi) + B_n \sin(n \phi)) \right] + r^{-n} (C_n \cos(n \phi) + D_n \sin(n \phi)) \]
\[
\phi(r, \theta) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[ a_n r^n \sin(n \theta + a_n) + b_n r^n \sin(n \theta + b_n) \right]
\]

If the region where there is no charge includes \( r = 0 \), then all \( b_n = 0 \) since \( \phi \) should not diverge at the origin. If \( r = 0 \) is excluded from the region, then the \( b_n \) need not be zero. The case \( b_0 \neq 0 \) corresponds to a line charge \( \lambda \) along the \( z \) axis.

Consider the case where \( \phi \) has a restricted range, for example a wedge shaped opening of angle \( \beta \)

\begin{align*}
0 &\leq \theta &\leq \beta \\
\phi \text{ is constant in conductor} &\Rightarrow \text{boundary conditions }
\end{align*}

\[
\begin{cases}
\phi(r, 0) = \phi_0 \\
\phi(r, \beta) = \phi_0
\end{cases}
\]

The general solution is the linear combination

\[
\phi(r, \theta) = (a_0 + b_0 \ln r)(A_0 + B_0 \phi) + \sum_{\nu > 0} \left( a_\nu r^\nu + b_\nu r^{-\nu} \right)(A_\nu \cos(\nu \theta) + B_\nu \sin(\nu \theta))
\]
1. The condition \( \phi(r, 0) = \phi_0 \) a constant independent of \( r \) then requires

\[
b_0 = 0, \quad a_0 = 0 \quad \text{all } \nu
\]

So

\[
\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{\nu > 0} (a_\nu r^\nu + b_\nu r^{-\nu}) B_\nu \sin(\nu \varphi)
\]

2. Since \( \phi \) should be continuous as one approaches the conducting surface, and \( \phi = \phi_0 \) is a finite constant on the conducting surface, then \( \phi \) cannot diverge as one approaches the origin \( r = 0 \) along any fixed angle \( \varphi \). This requires

\[
b_\nu = 0 \quad \text{all } \nu
\]

So

\[
\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{\nu > 0} a_\nu r^\nu \sin(\nu \varphi)
\]

3. The condition \( \phi(r, \beta) = \phi_0 \) a constant independent of \( r \) then requires

\[
\sin(\nu \beta) = 0 \implies \nu = \frac{n \pi}{\beta}, \quad n \text{ integer } \geq 1
\]

So

\[
\phi(r, \varphi) = a_0 (A_0 + B_0 \varphi) + \sum_{n=1}^\infty a_n r^{n \pi / \beta} \sin\left(\frac{n \pi \varphi}{\beta}\right)
\]

4. as \( \phi \) must approach the constant \( \phi_0 \) as \( r \to 0 \) along any fixed angle \( \varphi \), we therefore must have

\[
b_0 = 0, \quad a_0 A_0 = \phi_0
\]
So finally we have

\[ \phi(r, \varphi) = \phi_0 + \sum_{n=1}^{\infty} a_n r^\frac{n}{\beta} \sin\left(\frac{n\pi \varphi}{\beta}\right) \]

We still have all the unknowns \( a_n \)! These depend on how \( \phi(r, \varphi) \) behaves as \( r \to \infty \) (we can't make the choice here that \( \phi \to 0 \) as \( r \to \infty \)) - this is additional information that must be specified to find the complete solution.

Nevertheless we can still get very interesting information near the origin at small \( r \). In this case, the leading term in the above series expansion for \( \phi \) is the \( n=1 \) term, as it vanishes most slowly as \( r \to 0 \).

\[ \phi(r, \varphi) \approx \phi_0 + a_1 r^\frac{1}{\beta} \sin\left(\frac{\pi \varphi}{\beta}\right) \]

The electric field is

\[ E_r(r, \varphi) = -\frac{\partial \phi}{\partial r} = -\frac{\pi a_1}{\beta} r^\frac{1}{\beta} - 1 \sin\left(\frac{\pi \varphi}{\beta}\right) \]

\[ E_\varphi(r, \varphi) = -\frac{1}{r} \frac{\partial \phi}{\partial \varphi} = -\frac{\pi a_1}{\beta} r^\frac{1}{\beta} - 1 \cos\left(\frac{\pi \varphi}{\beta}\right) \]

\[ \Rightarrow E \sim r^\frac{1}{\beta} - 1 \]

**Induced surface charge given by**

\[ 4\pi \sigma = E \cdot \hat{n} \]
for surface at $\phi = 0$, $\hat{m} = \hat{n}$
for surface at $\phi = \beta$, $\hat{m} = -\hat{n}$

$$\sigma(r, \phi = 0) = \frac{E \phi(r, 0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta} - 1}$$

$$\sigma(r, \phi = \beta) = \frac{-E \phi(r, \beta)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta} - 1}$$

For $\frac{\pi}{\beta} > 1$, $0 < \beta < \pi$, $\tilde{E}$ and $\tilde{\sigma}$ vanish as
approach the origin.

For $\frac{\pi}{\beta} < 1$, $\pi < \beta < 2\pi$, $\tilde{E}$ and $\tilde{\sigma}$ diverge as
approach the origin.

$\beta = \frac{\pi}{4}$

$E \sim r^3$

$\beta = \pi$

$\tilde{E} \sim \text{const}$

$\beta = \frac{3\pi}{2}$

$E \sim r^{-\frac{1}{3}}$

$\beta = \frac{\pi}{2}$

$E \sim r$

$\beta = 2\pi$

$E \sim r^{-\frac{1}{2}}$

$E$ diverges at an "external" corner
$E$ vanishes at an "internal" corner

Remember, the above examples all had translational
symmetry along $z$, so the "corners" above are
really infinitely long straight "edges".