\[ \nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

\[ \phi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \]

\[ r^2 \nabla^2 \phi = \Theta \Phi \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R \Phi}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{R \Theta}{\sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \]

\[ \frac{r^2 \sin \theta}{\Phi} \nabla^2 \phi = \frac{\sin \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \]

\[ \text{depends only on } r \quad \text{ad } \theta \]
\[ \text{depends only on } \phi \]
\[ = -\text{const} \]
\[ = \text{const} \]

\[ \frac{1}{r^2} \frac{d^2 \Phi}{d\phi^2} = -m^2 \]

\[ \Rightarrow \Phi = e^{\pm i m \phi} \]

\[ \text{m integer for } 2\pi \text{ periodicity in } \phi \]

\[ \Rightarrow \frac{\sin \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = -m^2 \]

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0 \]

\[ \text{depends only on } r \]
\[ \text{depends only on } \theta \]
\[ = -\text{const} \]
\[ = -\text{const} \]
Call the constant \( L(l+1) \)

For \( R \):

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = L(l+1) = 0
\]

Solutions are of the form:

\[
R(r) = a_0 r^2 + b_0 r^{-(l+1)}
\]

Substitute \( u \) to verify:

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{d}{du} \left( r^2 (la_0 r^{e-1} - (l+1)b_0 r^{-e-2}) \right)
\]

\[
= \frac{d}{du} \left( la_0 r^{e+1} - (l+1)b_0 r^{-e} \right)
\]

\[
= l(l+1)a_0 r^2 + l(l+1)b_0 r^{-(l+1)} = L(l+1)R
\]

For \( \theta \):

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{d\theta}{d\phi}) - \frac{m^2}{\sin^2 \theta} = -L(l+1)
\]

Let \( x = \cos \theta \)

\[
dx = -\sin \theta \, d\theta
\]

\[
\frac{d\theta}{dx} = \frac{1}{\sin \theta}
\]

\( 0 \leq \theta \leq \pi \)

Solutions for \(-1 \leq x \leq 1\) correspond to \( L \geq 0 \) integers.

Above becomes:

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d\theta}{dx} \right] + \left[ L(l+1) - \frac{m^2}{1-x^2} \right] \theta = 0
\]

Called generalized Legendre Equation - solutions are called the associated Legendre functions. Ordinary Legendre polynomials are solutions for \( m = 0 \).
For the special case \( m = 0 \), i.e., the solution has azimuthal symmetry and \( \phi \) does not depend on the angle \( \theta \) (i.e., rotational symmetry about \( \phi \) axis),

we want the solutions to

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d\Theta}{dx} \right] + \ell(\ell+1) \Theta = 0
\]

The solutions are known as the Legendre polynomials, \( P_\ell(x) \).

They are given, for \( \ell \) integer, by

\[
P_\ell(x) = \frac{1}{2\ell+1} \frac{d^\ell}{dx^\ell} (x^2-1)^\ell
\]  

Rodriguez's formula

The lowest \( \ell \) polynomials are

\[
P_0(x) = 1, \quad P_2(x) = \frac{1}{2} (3x^2 - 1) \\
P_1(x) = x, \quad P_3(x) = \frac{1}{2} (5x^3 - 3x)
\]

In general, \( P_\ell(x) \) is a polynomial of order \( \ell \) with only even powers if \( \ell \) is even, and only odd powers if \( \ell \) is odd. \( \Rightarrow P_\ell(x) \) is even or \( x \) for \( \ell \) even \( \Rightarrow x \) for \( \ell \) odd

\( P_\ell(x) \) is normalized so that \( P_\ell(1) = 1 \).
Note: Legendre polynomials are only for integer \( l \geq 0 \). What about solutions for non-integer \( l \)?

The \( P_l(x) \) give one solution for each integer \( l \). But \( P_l(x) \) are defined by a 2nd order differential equation – shouldn't there be a 2nd independent solution for each \( l \)?

It turns out that these "2nd" solutions, as well as solutions for non-integer \( l \), all blow up at either \( x = -1 \) or \( x = 1 \), i.e., at \( \theta = 0 \) or \( \theta = \pi \). They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval \(-1 \leq x \leq 1\).

\[
\int_{-1}^{1} dx \ P_l(x) P_m(x) = \int_0^\pi \sin \theta \ P_l(\cos \theta) \ P_m(\cos \theta) = \left\{ \begin{array}{ll} 0 & \text{if } l \neq m \\ \frac{2}{2l+1} & \text{if } l = m \end{array} \right.
\]

"we can expand any function \( f(\theta) \), \( 0 \leq \theta \leq \pi \), as a linear combination of the \( P_l(\cos \theta) \). This is the reason they are useful for solving problems of Laplace's eqn with spherical boundary surfaces."
For \( m \neq 0 \), the solutions to (see Jackson 3.5)

\[
d \frac{1}{(1-x^2)} \frac{d}{dx} \left[ x \frac{d \Theta}{dx} \right] + \left[ \ell(l+1) - \frac{m^2}{1-x^2} \right] \Theta = 0
\]

are the associated Legendre functions \( P^m_\ell(x) \).

For \( P^m_\ell(x) \) to be finite in interval \(-1 \leq x \leq 1\),

one again finds that \( \ell \) must be integer \( \ell > 0 \), and integer \( m \) must satisfy \( |m| \leq \ell \), i.e. \( m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell \).

For each \( \ell \) and \( m \) there is only one such non-divergent solution.

It is typical to combine the solutions \( P^m_\ell(\cos \theta) \) to the \( \theta \)-part of the equation with the \( \Phi_m(\phi) = e^{im\phi} \) solutions to the \( \phi \)-part of the equation to define the spherical harmonics

\[
Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P^m_\ell(\cos \theta) e^{im\phi}
\]

The \( Y_{\ell m} \) are orthogonal

\[
\int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \sin \theta \quad Y^*_{\ell' m'}(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell \ell'} \delta_{m m'}
\]

and are a complete set of basis functions for expanding any function \( f(\theta, \phi) \) defined on the surface of a sphere.
Examples with azimuthal symmetry $m = 0$

General solution to $\nabla^2 \Phi = 0$ can be written in form

$$\Phi(r, \theta) = \sum_{\ell=0}^{\infty} \left[ A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos \theta)$$

determine the $A_\ell$ and $B_\ell$ from the boundary conditions of the particular problem.

1. Suppose one is given $\Phi(R, \theta) = \phi_0(\theta)$ on surface of sphere of radius $R$.

To find solution of $\nabla^2 \Phi = 0$ inside sphere

$\Phi$ should not diverge at origin $\Rightarrow B_\ell = 0$ for all $\ell$

$$\Phi(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos \theta)$$

$$\Rightarrow \Phi(r, \theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos \theta)$$

$$\Rightarrow \int_0^\pi d\theta \sin \theta \phi_0(\theta) P_m(\cos \theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi d\theta \sin \theta P_\ell(\cos \theta) P_m(\cos \theta)$$

$$= \sum_{\ell=0}^{\infty} A_\ell R^\ell \frac{\ell}{2\ell+1} g_{\ell m}$$

$$= A_m R^m \frac{\pi}{2m+1} \int_0^\pi d\theta \sin \theta \phi_0(\theta) P_m(\cos \theta)$$

$$A_m = \frac{2^{m+1}}{2^m} \int_0^\pi d\theta \sin \theta \phi_0(\theta) P_m(\cos \theta)$$
To find solution of $\Delta^2 \phi = 0$ outside sphere

If require $\phi \rightarrow 0$ as $r \rightarrow 0$, then $A_e = 0$ for all $l$

\[
\phi (r, \theta) = \sum_{l=0}^{\infty} \frac{B_e}{r^{l+1}} P_e (\cos \theta)
\]

\[
\phi (R, \theta) = \phi_0 (\theta) = \sum_{l=0}^{\infty} \frac{B_e}{R^{l+1}} P_e (\cos \theta)
\]

gives solution

\[
B_m = \frac{2^{m+1}}{2} R^{m+1} \int_0^\pi \sin \theta \phi_0 (\theta) P_m (\cos \theta) d\theta
\]

\[
B_m = A_m R^{2m+1}
\]

2) Suppose one is given surface charge density $\sigma (\theta)$ fixed on surface of sphere of radius $R$. What is $\phi$ inside and outside?

From previous example

\[
\phi (r, \theta) = \begin{cases} 
\sum_{l=0}^{\infty} A_e r^l P_e (\cos \theta) & r < R \\
\sum_{l=0}^{\infty} \frac{B_e}{r^{l+1}} P_e (\cos \theta) & r > R
\end{cases}
\]

Boundary conditions at $r = R$ on surface

(i) $\phi$ continuous

\[
\rightarrow \sum_{l=0}^{\infty} \left[ A_e R^l - \frac{B_e}{R^{l+1}} \right] P_e (\cos \theta) = 0
\]
If an expansion in Legendre polynomials vanishes for all θ, then each coefficient in the expansion must vanish.

\[ \Rightarrow A_e \frac{R^l}{R^{l+1}} = \frac{B_l}{R^{l+1}} \Rightarrow B_l = A_e R^{2l+1} \]

(cii) gpup in electric field at Ω:

\[ -\frac{\partial \Phi^\text{out}}{\partial r} \bigg|_{r=R} + \frac{\partial \Phi^\text{in}}{\partial r} \bigg|_{r=R} = 4\pi \sigma \]

\[ \Rightarrow \sum_{l=0}^{\infty} \left[ \frac{(l+1) B_l}{R^{l+2}} + l A_e R^{l-1} \int P_l(\cos \theta) = 4\pi \sigma \right] \]

\[ \Rightarrow \sum_{l=0}^{\infty} \left[ \frac{(2l+1) R^{l-1} A_e P_l(\cos \theta)}{R^{l+2}} \right] = 4\pi \sigma \]

\[ (2m+1) R^{m-1} A_m \left( \frac{2}{2m+1} \right) = 4\pi \int_0^\pi \sin \theta \sigma(\theta) P_m(\cos \theta) \]

\[ A_m = \frac{4\pi}{2 R^{m-1}} \int_0^\pi \sin \theta \sigma(\theta) P_m(\cos \theta) \]
Suppose \( \sigma(\theta) = k \cos \theta \)  

What is \( \phi \)?

Note: \( \sigma(\theta) = k P_1(\cos \theta) \)

hence only \( A_1 \neq 0 \) by orthogonality of \( P_n(\cos \theta) \)

\[
A_1 = \frac{4\pi k}{2} \int_0^\pi \sin \theta P_1(\cos \theta) P_1(\cos \theta) \, d\theta
\]

\[
= \frac{4\pi k}{2} \left( \frac{2}{2+1} \right) = \frac{4\pi k}{3}
\]

\[\Rightarrow \phi(r, \theta) = \begin{cases} 
\frac{4\pi k}{3} r \cos \theta & r < R \\
\frac{4\pi k}{3} \frac{R^3}{r^2} \cos \theta & r > R 
\end{cases}
\]

We will see that potential outside the sphere is that of an ideal dipole with dipole moment

\[ p = \frac{4\pi R^3 k}{3} \]

Inside the sphere, the potential \( \phi = \frac{4\pi k}{3} \zeta \)

where \( \zeta = r \cos \theta \). The electric field inside the sphere is therefore the constant

\[ \vec{E} = -\nabla \phi = -\frac{4\pi k}{3} \zeta \]
outside the sphere the field is

\[ E = -\nabla \phi = -\frac{\partial \phi}{\partial r} \mathbf{\hat{r}} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \mathbf{\hat{\theta}} \]

\[ = \frac{8 \pi k R^3}{3 \Gamma^3} \cos \theta \mathbf{\hat{r}} + \frac{4 \pi k R^3}{3 \Gamma^3} \sin \theta \mathbf{\hat{\theta}} \]

\[ \mathbf{E} = \frac{4 \pi R^3 k}{3 \Gamma^3} \left[ 2 \cos \theta \mathbf{\hat{r}} + \sin \theta \mathbf{\hat{\theta}} \right] \]

\[ \text{deplete field} \]
Physical example with $\sigma(\theta) = \kappa \cos \theta$.

Two spheres of radius $R$, with equal but opposite uniform charge densities $\rho$ and $-\rho$, displaced by small distance $d < R$.

Surface charge $\sigma$ builds up due to displacement. This is a uniformly "polarized" sphere.

Surface charge: $\sigma(\theta) = \rho \, Sr = \rho \, d \cos \theta$.

$\sigma(\theta) = \rho d \cos \theta$.

Total dipole moment is $(\rho d) \frac{4}{3} \pi R^3$.

Polarization = \frac{\text{dipole moment}}{\text{volume}} = \rho d$.

$E$ field inside a uniformly polarized sphere is constant. $E = -\rho d \frac{4\pi}{3}$. 

$E = -\rho d \frac{4\pi}{3}$. 

$E = -\rho d \frac{4\pi}{3}$.
\( \phi \) and \( \mathbf{E} \) at infinite distance from sphere. \( \mathbf{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 r \cos \theta \)

Boundary conditions:

\[ \begin{align*}
\phi (r, \theta) &= 0 \\
\phi (r \to \infty, \theta) &= -E_0 r \cos \theta
\end{align*} \]

Solution outside sphere has the form:

\[ \phi (r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l (\cos \theta) \]

From boundary condition as \( r \to \infty \), we have:

\[ A_0 = 0 \quad \text{all } l \neq 1 \]

\[ A_1 = -E_0 \quad \text{since } P_1 (\cos \theta) = \cos \theta \]

\[ \phi (r, \theta) = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l (\cos \theta) \]

From \( \phi (r, \theta) = 0 \) we have:

\[ 0 = -E_0 r \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l (\cos \theta) \]

\[ \Rightarrow B_l = 0 \quad \text{all } l \neq 1 \]

\[ B_1 = E_0 R \Rightarrow B_1 = +E_0 R^3 \]
\[ \phi(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos \theta \]

1st term is just potential \(-E_0 r \cos \theta\) of the uniform applied electric field.

2nd term is potential due to the induced surface charge on the surface—it is a dipole field.

Induced charge density is

\[ \frac{4\pi \sigma(\theta)}{\partial r} \bigg|_{r=R} = E_0 \left( 1 + \frac{2R^3}{r^3} \right) \cos \theta \]

\[ = 3E_0 \cos \theta \]

\[ \sigma(\theta) = \frac{3}{4\pi} E_0 \cos \theta \quad \text{like uniformly polarized sphere} \]

From (2) we know that the field inside the sphere due to this \( \sigma \) is just

\[ -\frac{1}{3} \mu \kappa 2 = -\frac{1}{3} \mu 3E_0 \frac{\dot{\phi}}{4\pi} \]

\[ = -E_0 \dot{\phi} \]. This is just what is required so that the total field in the conducting sphere vanishes.

Can check that outside the sphere, \( \vec{E} = -\nabla \phi \)

is normal to surface of sphere at \( r = R \).
Behavior of fields near crucial hole or sharp tip

We now want to solve the $\nabla^2 \phi = 0$

with separation of variables,

but now $\Theta$ is restricted to range

$0 \leq \Theta < \beta$.

We still have azimuthal symmetry,

but now, since we do not need solution to $\phi$ be finite

for all $\Theta \in [0, \pi]$, but only $\Theta \in (0, \beta)$, we have more

solutions to the $\Theta$ equation, i.e. it does not have to

be integer, still need $l > 0$ to be finite at $\Theta = 0$.

See Jackson sec. 3.4 for details.