Boundary value problems in magnetostatics
Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{J}$$

and the fact that $\nabla^2 \vec{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar $-\nabla^2 \phi = 4\pi\rho$ dont work so well for magnetostatics.

However, in situations where the current $\vec{J}$ is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential ϕ_M.

In regions where $\vec{J} = 0$, ie not on the certain surfaces we have $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. Since $\vec{\nabla} \times \vec{B} = 0$ in these regions we can define a scalar potential ϕ_M such that

$$\vec{B} = -\vec{\nabla} \phi_M$$

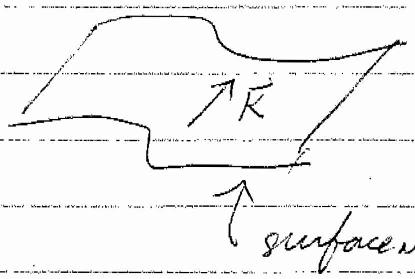
and then

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 \phi_M = 0$$

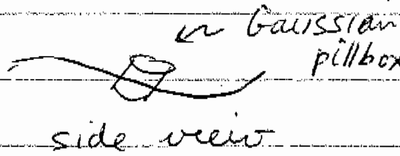
We can solve for ϕ_M as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

Bounday Conditions at Sheet current

in magnetostatics $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$



surface current $\vec{K}(\vec{r})$ at pt $\vec{r}$ on surface S

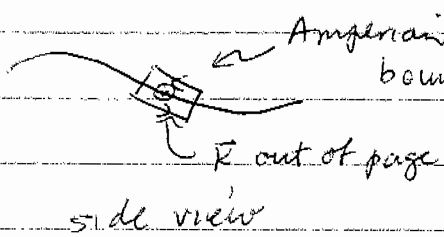


$$\int_V d^3r\ \vec{\nabla}\cdot\vec{B} = 0$$

top + bottom area of pill box is da
width of pill box $\to 0$

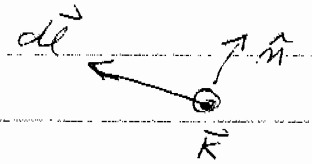
$$\Rightarrow \int_V d^3r\ \vec{\nabla}\cdot\vec{B} = \oint_S da\ \hat{n}\cdot\vec{B} = da\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot\hat{n} = 0$$

normal component of $\vec{B}$ is continuous $\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot\hat{n} = 0$



$$\int_S da\ \hat{n}\cdot(\vec{\nabla}\times\vec{B}) = \oint_C \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I_{enclosed}$$

let width of loop $\to 0$, top + bottom sides $d\vec{l}$



$(\hat{n}\times d\vec{l})\cdot\vec{K}$

$$\left(\vec{B}_{above} - \vec{B}_{below}\right)\cdot d\vec{l} = \frac{4\pi}{c}$$
$$= \frac{4\pi}{c}\left(\vec{K}\times\hat{n}\right)\cdot d\vec{l}$$

$\hat{n}$ is outward normal

tangential component of $\vec{B}$ has discontinuous jump $\frac{4\pi}{c}\vec{K}\times\hat{n}$

Combine both results into

$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \frac{4\pi}{c} \vec{K} \times \hat{n}}$$

magnetic analog of $\vec{E}_{above} - \vec{E}_{below} = 4\pi\sigma \hat{n}$

In terms of magnetic ~~vector~~ scalar potential ϕ_M

$$-\vec{\nabla}\phi_{M\,above} + \vec{\nabla}\phi_{M\,below} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note: ϕ_M is a calculational tool only
it does not have any direct physical
significance as does the electrostatic ϕ.
Electrostatic ϕ is related to work done
moving a charge $W_{12} = q[\phi(r_2) - \phi(r_1)]$
nothing similar for ϕ_M.
(in fact ~~magnetostatic~~ magnetic forces do no work!

$$\vec{F} = q\, \vec{v} \times \vec{B}$$

$$\Rightarrow \vec{F} \cdot \vec{v} = \frac{dW}{dt} = 0 \;)$$

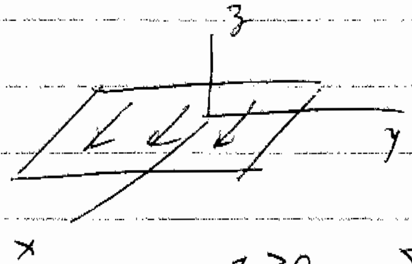
Note: ~~We cannot apply argument like $\phi(r) - \phi(r') = \int_{r'}^{r} \vec{E} \cdot d\vec{l}$~~
ϕ_M is not necessarily continuous at surface current
Cannot do similar to electrostatics and use

$$\phi_M(r_{above}) - \phi_M(r_{below}) = -\int_{r_{below}}^{r_{above}} \vec{B} \cdot d\vec{l}$$

Since ϕ_M is not defined on the current sheet
itself, separating "above" from "below".

example

Flat infinite plane at $z=0$ with surface current $\vec{K} = K\hat{x}$



$$z>0, \quad \nabla^2\Phi_M^> = 0 \Rightarrow \Phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$
$$z<0, \quad \nabla^2\Phi_M^< = 0 \Rightarrow \Phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z>0, \quad \vec{B}^> = -\vec{\nabla}\Phi_M^> = b_x^>\hat{x} + b_y^>\hat{y} + b_z^>\hat{z}$$
$$z<0, \quad \vec{B}^< = -\vec{\nabla}\Phi_M^< = b_x^<\hat{x} + b_y^<\hat{y} + b_z^<\hat{z}$$

at $z=0$ $\quad \vec{B}^> - \vec{B}^< = (b_x^> - b_x^<)\hat{x} + (b_y^> - b_y^<)\hat{y} + (b_z^> - b_z^<)\hat{z}$
$$= \frac{4\pi}{c}\vec{K}\times\hat{n} = \frac{4\pi}{c}K(\hat{x}\times\hat{z}) = -\frac{4\pi}{c}K\hat{y}$$

$$\Rightarrow b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi}{c}K$$

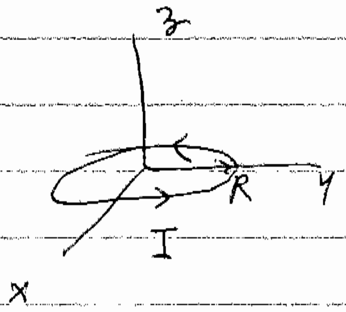
define $\left.\begin{array}{l} b_y^> = b_{y0} + \delta b_y \\ b_y^< = b_{y0} - \delta b_y \end{array}\right\} \delta b_y = -\frac{2\pi}{c}K$

$$\Rightarrow \vec{B}^> = \vec{B}_0 - \frac{2\pi}{c}K\hat{y} \qquad \vec{B}_0 = b_{x0}\hat{x} + b_{y0}\hat{y} + b_{z0}\hat{z}$$
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi}{c}K\hat{y}$$

if $\vec{K}$ is the only source of magnetic field then $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi}{c}K\hat{y} & z>0 \\ \frac{2\pi}{c}K\hat{y} & z<0 \end{cases}$$

example circular current loop in xy plane
radius R



for $r > R$, $\vec{\nabla}\times\vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla}\phi_M$
where $\nabla^2\phi_M = 0$.

Try Legendre polynomial expansion for ϕ_M

$$\phi_M = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad (A_\ell \text{ terms vanish as want } B \to 0 \text{ as } r \to \infty)$$

$$\vec{B} = -\vec{\nabla}\phi_M = -\frac{\partial\phi_M}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi_M}{\partial\theta}\hat{\theta}$$

$$= \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\,\hat{r} - \frac{B_\ell}{r^{\ell+2}} \frac{\partial P_\ell(\cos\theta)}{\partial\theta}\,\hat{\theta} \right]$$

write $\frac{\partial P_\ell}{\partial\theta} = \frac{\partial P_\ell}{\partial x}\frac{\partial x}{\partial\theta} = -\frac{\partial P_\ell}{\partial x}\sin\theta \qquad x = \cos\theta$

$$= -P_\ell'\sin\theta$$

$$\vec{B} = \sum_\ell \left[\frac{(\ell+1)B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)\,\hat{r} + \frac{B_\ell}{r^{\ell+2}} \sin\theta\, P_\ell'(\cos\theta)\,\hat{\theta} \right]$$

To determine the B_ℓ we compare with exact solution along $\hat{z}$ axis

$$\vec{B}(z\hat{z}) = \sum_\ell \frac{(\ell+1)B_\ell}{r^{\ell+2}}\hat{r} = \sum_\ell \frac{(\ell+1)B_\ell}{z^{\ell+2}}\hat{z}$$

since $P_\ell(1) = 1$, $\sin(0) = 0$ and $P_\ell'(1)$ finite, $\hat{r} = \hat{z}$ when $\theta = 0$

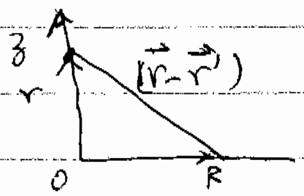
exact solution on $\hat{z}$ axis:

$$\vec{A} = \int \frac{d^3r'}{c} \frac{\vec{J}(r')}{|\vec{r}-\vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A} = \int \frac{d^3r'}{c} \vec{\nabla}\times \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{B} = -\int \frac{d^3r'}{c} \vec{J}(\vec{r}') \times \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$\vec{B} = \int \frac{d^3r'}{c} \vec{J}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$ Biot-Savart Law for magnetostatics

For our loop



polar radial vector

$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c} I\hat{\phi} \times \frac{[-R\hat{r} + z\hat{z}]}{(z^2+R^2)^{3/2}}$$

$\hat{r}\times\hat{\phi} = \hat{z}$

$$= \int_0^{2\pi} \frac{d\phi}{c} \frac{R(IR)\hat{z}}{(z^2+R^2)^{3/2}}$$ $\hat{\phi}\times\hat{z}$ term integrates to zero

$$\boxed{\vec{B}(z) = \frac{2\pi R^2 I\,\hat{z}}{c(z^2+R^2)^{3/2}}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I\,\hat{z}}{c z^3} \frac{1}{\left(1+\left(\frac{R}{z}\right)^2\right)^{3/2}} = \frac{2\pi R^2 I\,\hat{z}}{z^3}\left\{1 - \frac{3}{2}\left(\frac{R}{z}\right)^2 + \dots\right\}$$

$$= \frac{2\pi R^2 I\,\hat{z}}{c}\left\{\frac{1}{z^3} - \frac{3}{2}\frac{R^2}{z^5} + \dots\right\}$$

$$= \left\{\frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \dots\right\}\hat{z}$$

$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3}{4c}\pi R^2 I R^2$$

So to order $l = 3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2P_1(\cos\theta)\hat{r} + \sin\theta P_1'(\cos\theta)\hat{\theta}}{r^3} - \left[\frac{3R^2 P_3(\cos\theta)\hat{r} + \frac{3}{4}R^2 \sin\theta P_3'(\cos\theta)\hat{\theta}}{r^5}\right] + \cdots \right\}$$

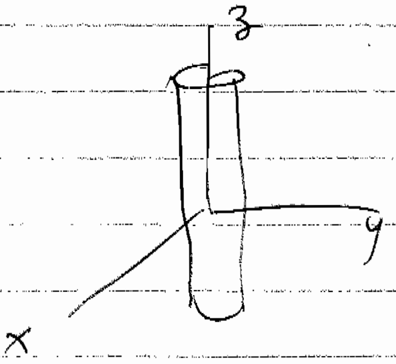
$$P_1(x) = x \Rightarrow P_1'(x) = 1$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \tfrac{1}{2}(15x^2 - 3)$$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c}\left\{ \frac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3} - \left[\frac{\frac{3}{2}R^2(5\cos^3\theta - 3\cos\theta)\hat{r} + \frac{3}{8}R^2\sin\theta(15\cos^2\theta - 3)\hat{\theta}}{r^5}\right] + \cdots \right.$$

$\frac{\pi R^2 I}{c} = m$ is the magnetic dipole moment of the loop

We see that the 1^{st} term is just the magnetic dipole approx. The 2^{nd} term is the magnetic ~~quadrupole~~ octopole term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5-40) example current carrying infinite cylinder radius R



(i) $\vec{K} = K\hat{z}$ wire with surface current

(ii) $\vec{K} = K\hat{\varphi}$ solenoid

(i) $\vec{K} = K\hat{z}$ $\qquad 2\pi R K = I$ total current

"guess" + show it is correct

$r > R$ $\quad \boxed{\Phi_M = -\frac{4\pi}{c} R K \varphi}$ $\quad$ magnetic scalar potential $\quad \nabla^2\Phi_M = 0$

$r < R$ $\quad \Phi_M = 0$

$r > R$ $\quad \vec{B} = -\vec{\nabla}\Phi_M = -\frac{1}{r}\frac{\partial \Phi_M}{\partial \varphi}\hat{\varphi} = \frac{4\pi R K}{c r}\hat{\varphi} = \boxed{\frac{2I}{cr}\hat{\varphi}}$ familiar result from Ampere

$r < R$ $\quad \vec{B} = 0$

$$\vec{B}_{above} - \vec{B}_{below} = \frac{2I}{cR}\hat{\varphi} = \frac{4\pi K}{c}\frac{R}{R}\hat{\varphi} = \frac{4\pi}{c}\vec{K} \times \hat{n}$$

where $\hat{n} = \hat{r}$

as $\hat{z} \times \hat{r} = \hat{\varphi}$

Note: $\Phi_M = -\frac{4\pi}{c} R K \varphi$ is not single valued.!
would not have found this using expansion
of separation of coords in polar coords

(ii) $\vec{K} = K\hat{\varphi}$

$r > R$ $\quad \Phi_M = -B_1 z$

$r < R$ $\quad \Phi_M = -B_2 z$ $\quad \Big\}\ \nabla^2\Phi_M = 0$

$r > R$ $\quad \vec{B} = -\vec{\nabla}\Phi_M = B_1\hat{z}$

$r < R$ $\quad \vec{B} = -\vec{\nabla}\Phi_M = B_2\hat{z}$

$$\vec{B}_{above} - \vec{B}_{below} = (B_1 - B_2)\,\hat{z} = \frac{4\pi}{c}\vec{K} \times \hat{n}$$

$$= \frac{4\pi}{c}K\,(\hat{\varphi} \times \hat{r})$$

$$= -\frac{4\pi}{c}K\,\hat{z}$$

If current in solenoid is only source of $\vec{B}$ then expect $B_1 = 0$

$$\Rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c}K\,\hat{z}}$$ familiar result