PHY 415

lecture 14

Define macroscopic current density

$$\vec{j}(\vec{r},t) = \Big\langle \sum_{i \in \text{free}} q_i \vec{v}_i \delta(\vec{r}-\vec{r}_i) \Big\rangle + \Big\langle \sum_{\substack{n \\ \text{molecs}}} q_n \vec{v}_n \delta(\vec{r}-\vec{r}_n) \Big\rangle$$

current of free charges — current of molecular drifting if molecules are charged

Then $\langle \vec{j}_o \rangle = \vec{j} + c\, \vec{\nabla} \times \vec{M} + \dfrac{\partial \vec{P}}{\partial t}$

Ampere's law becomes upon averaging

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \langle \vec{j}_o \rangle + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$= \frac{4\pi}{c} \vec{j} + 4\pi \vec{\nabla} \times \vec{M} + \frac{4\pi}{c} \frac{\partial \vec{P}}{\partial t} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times (\vec{B} - 4\pi \vec{M}) = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} + 4\pi \vec{P})$$

define $\boxed{\vec{H} \equiv \vec{B} - 4\pi \vec{M}}$ to get

$$\boxed{\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}}$$

$\boxed{\vec{D} = \vec{E} + 4\pi \vec{P}}$ as before

official nomenclature: $\vec{B}$ is the magnetic induction

$\vec{H}$ is the magnetic field

common usage: both $\vec{H}$ and $\vec{B}$ are called magnetic field when atoms have intrinsic magnetic moments due to electron spin, we can add these to $\vec{M}$ in obvious way

when molecules are neutral, $q_n = 0$, the "bound current" is given by

$$\vec{j}_{bound} = \sum_n \langle \vec{j}_n \rangle = c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of <u>bound charge</u>

$$\vec{\nabla}\cdot\vec{j}_{bound} = c\,\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{M}\right) + \vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t}$$

$$= 0 + \frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{P}\right)$$

$$= -\frac{\partial \rho_{bound}}{\partial t}$$ where $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$ is bound charge density

So $$\boxed{\vec{\nabla}\cdot\vec{j}_{bound} + \frac{\partial \rho_{bound}}{\partial t} = 0}$$

and bound charge is conserved.

Since total average charge must be conserved, ie

$$\vec{\nabla}\cdot\langle \vec{j}_0 \rangle - \frac{\partial \langle \rho_0 \rangle}{\partial t} = 0,$$ and $\langle \vec{j}_0 \rangle = \vec{j} + \vec{j}_{bound}$

free current

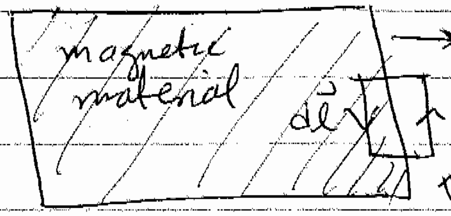
$\langle \rho_0 \rangle = \rho + \rho_{bound}$

free charge

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = 0}$$

free charge is also conserved

## At a surface of a magnetic material




$\rightarrow \hat{n}$ outward normal to surface

take $\hat{z} \equiv \hat{dl} \times \hat{n}$ out of page

Amperean loop C bounding surface S of area $da$

$$c \int_S da\, \hat{z} \cdot (\vec{\nabla} \times \vec{M}) = \int_S da\, \hat{z} \cdot \vec{J}_{bound} = da\, \hat{z} \cdot \vec{J}_{bound}$$

$$= (\vec{dl} \times \hat{n}) \cdot \vec{K}_{bound} \quad \text{as width of loop} \to 0$$

$$= (\hat{n} \times \vec{K}_{bound}) \cdot \vec{dl}$$

But by Stokes theorem

$$c \int_S da\, \hat{z} \cdot (\vec{\nabla} \times \vec{M}) = c \int_C \vec{dl} \cdot \vec{M} = c\, \vec{dl} \cdot \vec{M}$$

since width $\to 0$ and $\vec{M} = 0$ outside

$$\Rightarrow c\, \vec{dl} \cdot \vec{M} = (\hat{n} \times \vec{K}_{bound}) \cdot \vec{dl}$$

for any $\vec{dl}$ in plane of surface

$$\Rightarrow c\vec{M}_t = \hat{n} \times \vec{K}_{bound}$$

where $\vec{M}_t$ is component of $\vec{M}$ tangential to the surface (since $\vec{K}_b$ is in plane of surface, $\hat{n} \times \vec{K}$ is also entirely in the plane of the surface)

$$\Rightarrow c\, \hat{n} \times \vec{M}_t = c\, \hat{n} \times \vec{M} = \hat{n} \times (\hat{n} \times \vec{K}_{bound})$$

$$= -\vec{K}_{bound}$$

$$\Rightarrow \boxed{\begin{aligned} \vec{K}_{bound} &= c\, \vec{M} \times \hat{n} \\ \vec{J}_{bound} &= c\, \vec{\nabla} \times \vec{M} \end{aligned}}$$

## Total bound charge vanishes

$$Q_{bound} = \int_V d^3r\, \rho_{bound} + \int_S da\, \sigma_{bound}$$

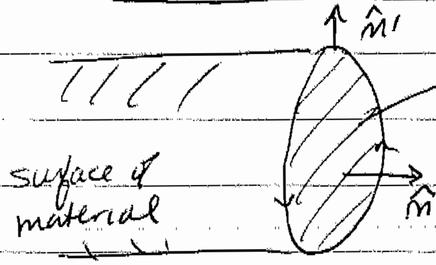
$\uparrow$ vol of dielectric $\quad \nwarrow$ surface of dielectric

$$= \int_V d^3r \, -\vec{\nabla}\cdot\vec{P} + \int da\, \hat{n}\cdot\vec{P}$$

but by Gauss theorem $\int_V d^3r\, \vec{\nabla}\cdot\vec{P} = \int da\, \hat{n}\cdot\vec{P}$

so $Q_{bound} = -\int da\, \hat{n}\cdot\vec{P} + \int da\, \hat{n}\cdot\vec{P} = 0$

## Total bound current vanishes




crossectional area of magnetic material

$S$ - bounding curve is $C$

$\hat{n}$ is normal to crossecti[on]
$\hat{n}'$ is normal to surface

total current flowing through $S$ is

$$\int_S da\, \hat{n}\cdot\vec{J}_{bound} + \int_C dl\, \vec{K}_{bound}\cdot\hat{n}$$

$$= c\int_S da\, \hat{n}\cdot(\vec{\nabla}\times\vec{M}) + c\int_C dl\, \hat{n}\cdot(\vec{M}\times\hat{n}')$$

$$= c\int_C d\vec{l}\cdot\vec{M} + c\int_C dl\, \underbrace{(\hat{n}'\times\hat{n})}_{= -\hat{t}}\cdot\vec{M}$$

$= -\hat{t}$ unit tangent, $d\vec{l} = dl\,\hat{t}$

$$= c\int_C d\vec{l}\cdot\vec{M} - c\int_C d\vec{l}\cdot\vec{M} = 0$$