# Macroscopic Maxwell Equations

$\vec{\nabla}\cdot\vec{B} = 0$ $\qquad\qquad$ $\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$ $\qquad\qquad$ $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$

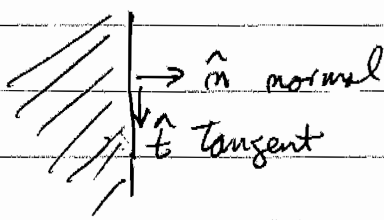
where $\rho$ and $\vec{j}$ are macroscopic charge + current densities
do not include bound charges or currents

$\vec{D} = \vec{E} + 4\pi\vec{P}$ , $\vec{P}$ is polarization density

$\vec{H} = \vec{B} - 4\pi\vec{M}$ , $\vec{M}$ is magnetization density

## Boundary conditions for statics

electrostatics: at surface of a dielectric, or at interface between two different dielectrics




$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \hat{t}\cdot\vec{E}_{above} = \hat{t}\cdot\vec{E}_{below}$

tangential component $\vec{E}$ is continuous

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \Rightarrow \hat{n}\cdot(\vec{D}_{above} - \vec{D}_{below}) = 4\pi\sigma$

normal component of $\vec{D}$ jumps by $4\pi\sigma$

magnetostatics: at surface or interface of magnetic materials

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \hat{n}\cdot\vec{B}_{above} - \hat{n}\cdot\vec{B}_{below}$

normal component of $\vec{B}$ is continuous

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} \Rightarrow \hat{t}\cdot(\vec{H}_{above} - \vec{H}_{below}) = \frac{4\pi}{c}\vec{K}$

tangential component of $\vec{H}$ jumps by $\frac{4\pi}{c}\vec{K}$

if $\sigma = 0$, ie no free surface charge, then $\hat{n}\cdot\vec{D}$ continuous

if $\vec{K} = 0$, ie no free surface current, then $\hat{t}\cdot\vec{H}$ continuous

## Examples

① Uniformly polarized sphere of radius $R$ $\quad \vec{\mathbb{P}} = \mathbb{P}\hat{z}$

$\vec{\mathbb{P}} \uparrow$ $\quad$ bound charge $\quad \rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}} = 0$ as $\vec{\mathbb{P}}$ constant

$$\sigma_b = \hat{n}\cdot\vec{\mathbb{P}} = \hat{r}\cdot\vec{\mathbb{P}} = \mathbb{P}\cos\theta$$

we saw earlier that a sphere with surface charge $\sigma(\theta) = \sigma_0 \cos\theta$ gives an electric field like a pure dipole for $r \geq R$, and is constant for $r < R$.

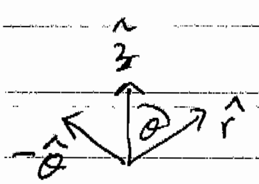
$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 \mathbb{P}\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ -\dfrac{4\pi\mathbb{P}}{3}\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{p} = \frac{4}{3}\pi R^3 \vec{\mathbb{P}}$

check behavior at boundary

Tangential component $\vec{E}$

$$\vec{E}^{\,t}_{above} = \left(\tfrac{4}{3}\pi R^3 \mathbb{P}\right)\frac{\sin\theta\,\hat{\theta}}{R^3} = \frac{4\pi}{3}\mathbb{P}\sin\theta\,\hat{\theta}$$




$$\vec{E}^{\,t}_{below} = -\frac{4\pi\mathbb{P}}{3}(\hat{z}\cdot\hat{\theta})\,\hat{\theta} = \frac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$

$\Rightarrow$ Tangential component $\vec{E}$ is continuous

normal component of $\vec{D}$

outside: $\vec{\mathbb{P}} = 0 \Rightarrow \vec{D} = \vec{E}$

$$\Rightarrow \hat{n}\cdot\vec{D} = \hat{r}\cdot\vec{E} = \left(\tfrac{4}{3}\pi R^3\mathbb{P}\right)\frac{2\cos\theta\,\hat{r}}{R^3} = \frac{8}{3}\pi\,\mathbb{P}\cos\theta\,.$$

inside: $\vec{E} = -\frac{4\pi \vec{P}}{3} \Rightarrow \vec{P} = -\frac{3}{4\pi}\vec{E}$

$$\vec{D} = \vec{E} + 4\pi\vec{P} = \vec{E} - 3\vec{E} = -2\vec{E} = \frac{8\pi}{3} P \hat{z}$$

$$\hat{n}\cdot\vec{D} = \hat{r}\cdot\left(\frac{8\pi}{3} P \hat{z}\right) = \frac{8\pi}{3} P \cos\theta$$

$\Rightarrow$ normal component $\vec{D}$ is continuous

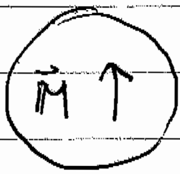
Note: normal component of $\vec{E}$ should jump by $4\pi\sigma_b = 4\pi P\cos\theta$

to check this: $\hat{n}\cdot\vec{E}^{\text{inside}} = \hat{r}\cdot\left(-\frac{4}{3}\pi P\hat{z}\right) = -\frac{4}{3}\pi P\cos\theta$

$$\hat{n}\cdot\left(\vec{E}^{\text{above}} - \vec{E}^{\text{below}}\right) = \frac{8}{3}\pi P\cos\theta + \frac{4}{3}\pi P\cos\theta$$

$$= \frac{12}{3}\pi P\cos\theta = 4\pi P\cos\theta \overset{4\pi}{=} \sigma_b(\theta)$$

(2) Uniformly magnetized sphere of radius $R$ $\quad \vec{M} = M\hat{z}$




bound current $\quad \vec{J}_b = c\vec{\nabla}\times\vec{M} = 0$ as $\vec{M}$ constant

$$\vec{K}_b = c\vec{M}\times\hat{n} = cM(\hat{z}\times\hat{r})$$
$$= cM\sin\theta\,\hat{\varphi}$$

we saw earlier that a sphere with surface current $\vec{K}_b = K_0\sin\theta\,\hat{\theta}$ gives a magnetic field that is pure dipole for $r > R$, and is constant for $r < R$.

$$\vec{B}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 M\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ \frac{8}{3}\pi M\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$

check behavior at boundary

normal component of $\vec{B}$

$$\hat{n}\cdot\vec{B}^{\,above} = \hat{r}\cdot\vec{B}^{\,above} = \frac{8}{3}\pi M\cos\theta$$

$$\hat{n}\cdot\vec{B}^{\,below} = \hat{r}\cdot\vec{B}^{\,below} = \frac{8}{3}\pi M(\hat{r}\cdot\hat{z}) = \frac{8}{3}\pi M\cos\theta$$

$\Rightarrow$ normal component of $\vec{B}$ is continuous

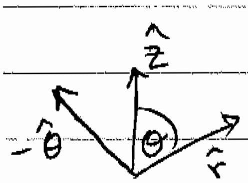
tangential component of $\vec{H}$

outside: $\vec{M}=0 \Rightarrow \vec{H}=\vec{B}$

$$\vec{H}^{t}_{above} = \left(\frac{4}{3}\pi M\right)\sin\theta\,\hat{\theta}$$

inside: $$\vec{H} = \vec{B} - 4\pi\vec{M} = \vec{B} - 4\pi\left(\frac{3}{8\pi}\vec{B}\right) = \vec{B} - \frac{3}{2}\vec{B} = -\frac{1}{2}\vec{B}$$

$$= -\frac{4\pi}{3}M\hat{z}$$




so $$\vec{H}^{t}_{below} = -\frac{4\pi}{3}M(\hat{z}\cdot\hat{\theta}) = \frac{4\pi}{3}M\sin\theta\,\hat{\theta}$$

$\Rightarrow$ tangential component $\vec{H}$ is continuous

Note: tangential component $\vec{B}$ should jump by $\frac{4\pi}{c}\vec{K}_b = 4\pi M\sin\theta\,\hat{\theta}$

inside:

to check: $$\vec{B}^{t}_{below} = \frac{8}{3}\pi M(\hat{z}\cdot\hat{\theta})\hat{\theta} = -\frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$\vec{H}^{t}_{above} = \vec{B}^{t}_{above} \Rightarrow$ $$\vec{B}^{t}_{above} - \vec{B}^{t}_{below} = \frac{4}{3}\pi M\sin\theta\,\hat{\theta} + \frac{8}{3}\pi M\sin\theta\,\hat{\theta}$$

$$= 4\pi M\sin\theta\,\hat{\theta} = \frac{4\pi}{c}\vec{K}_b$$

# Linear Materials

## Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

where $\rho$ and $\vec{j}$ are macroscopic charge + current densities and

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \vec{P} \text{ is polarization density}$$
$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad \vec{M} \text{ is magnetization density}$$

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In some materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

Ferromagnet: $\vec{M}$ can be non zero even if $\vec{B} = 0$

Ferroelectric: $\vec{P}$ can be non zero even if $\vec{E} = 0$

But more common are linear materials in which, for small $\vec{E}$ and $\vec{B}$, one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

linear dielectric

$\vec{P} = \chi_e \vec{E}$ $\quad$ $\chi_e$ is "electric susceptibility"

$\chi_e > 0$ for statics

$\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi \chi_e) \vec{E}$

$\vec{D} = \epsilon \vec{E}$ with $\epsilon = 1 + 4\pi \chi_e$

$\epsilon$ is the dielectric constant

linear magnetic materials

$\vec{M} = \chi_m \vec{H}$ $\quad$ $\chi_m$ is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \chi_m \vec{H}$

$\vec{B} = (1 + 4\pi \chi_m) \vec{H}$

$\vec{B} = \mu \vec{H}$ with $\mu = 1 + 4\pi \chi_m$

$\mu$ is magnetic permeability

For statics, $\chi_e > 0$ and $\chi_m$ (or alternatively $\epsilon$ and $\mu$) are constants depending on the material.

When we consider dynamics we will see that $\epsilon$ becomes a function of frequency.

## Clausius - Mossotti equation
Electric susceptibility + atomic polarizability

If a field $\vec{E}_{loc}$ is applied to an atom, it gets polarized

$$\vec{p} = \alpha \vec{E}_{loc}$$

"local field" - field the atom sees

atomic dipole moment

atomic polarizability

$\alpha$ is what one calculates from a microscopic theory

If $\vec{E}_{loc} = \vec{E}$ the average field in the material then electric susceptibility given by

$$\vec{\mathbb{P}} = n\vec{p} = n\alpha\vec{E}_{loc} = n\alpha\vec{E} = \chi_e \vec{E}$$

$\Rightarrow \chi_e = n\alpha$ where $n$ = density of atoms

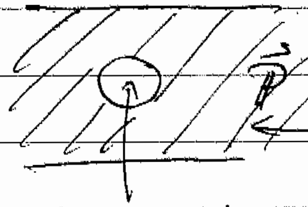
But a more careful consideration shows $\vec{E}_{loc} \neq \vec{E}$. The average field $\vec{E}$ includes the electric field created by the polarized atom itself. $\vec{E}_{loc}$, the local field the atom sees, should exclude its own self field.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

average field

average field excluding atom

average field of the atom




cut out sphere whose volume is $1/n$
the volume per atom

$\vec{E}_{loc}$ is field excluding the field of the polarized sphere of volume $1/n$.

$\vec{E}_{atom}$ is field of the polarized sphere

$$\vec{E}_{atom} = -\frac{4\pi\vec{\mathbb{P}}}{3} = -\frac{4\pi}{3} n\vec{p}$$

$$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi}{3}\vec{\mathbb{P}} = \vec{E} + \frac{4\pi}{3} n\vec{p}$$

$$\vec{p} = \alpha\vec{E}_{loc} = \alpha\left(\vec{E} + \frac{4\pi}{3} n\vec{p}\right) = \alpha\vec{E} + \frac{4\pi n\alpha}{3}\vec{p}$$

$$\vec{p} = \frac{\alpha}{1 - \frac{4\pi}{3} n\alpha}\vec{E}$$

$$\vec{\mathbb{P}} = n\vec{p} = \frac{\alpha n}{1 - \frac{4\pi}{3} n\alpha}\vec{E} = \chi_e\vec{E}$$

$$\boxed{\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3} n\alpha}}$$

or solve for $\alpha$ in terms of $\epsilon$

$$\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3} n\alpha} \Rightarrow \chi_e - \frac{4\pi}{3} n\alpha\chi_e = \alpha n$$

$$\Rightarrow \alpha = \frac{\chi_e}{n\left(1 + \frac{4\pi}{3}\chi_e\right)}$$

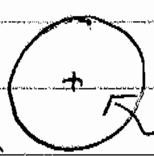
$$\epsilon = 1 + 4\pi\chi_e \Rightarrow \alpha = \frac{\epsilon - 1}{4\pi n} \frac{1}{\left(1 + \frac{\epsilon - 1}{3}\right)}$$

relates atomic polarizability to measured dielectric constant

$$\boxed{\alpha = \frac{3}{4\pi n}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)}$$

Clausius Mossotti or Lorentz-Lorenz equation

simple model for $\alpha$



uniform $\rho = \frac{-q}{\frac{4}{3}\pi a^3}$

atomic radius $a$

field inside is $E(r) = \frac{4\pi\rho}{3} r \hat{r}$

~~polarization~~

In external field $E_0$, net forces balance $\Rightarrow qE_0 = q\frac{4\pi\rho d}{3}$

$$p = qd = \frac{3}{4\pi\rho} q E_0 = \frac{3\left(\frac{4}{3}\pi a^3\right)}{4\pi q} q E_0$$

$$= a^3 E_0 \Rightarrow \alpha = a^3$$

$$\chi_e = \frac{n a^3}{1 - \frac{4\pi}{3} n a^3}$$

if $f = n\frac{4}{3}\pi a^3$ fraction of vol that is occupied by atoms

$$\boxed{\chi_e = \frac{1}{4\pi}\frac{3f}{1-f}}$$