Momentum Conservation

For charges \( q_i \) at positions \( \vec{r}_i \) with velocities \( \vec{v}_i \),

\[
\frac{d \vec{P}_{\text{mech}}}{dt} = \sum_i \frac{d \vec{P}_i}{dt} = \sum_i q_i \left( \vec{E}(\vec{r}_i) + \frac{1}{c^2} \vec{v}_i \times \vec{B}(\vec{r}_i) \right)
\]

"Mechanical" force on

\[ \text{momentum of charge} \]

\[
\vec{p}_E + \frac{1}{c^2} \vec{v} \times \vec{B} = \frac{1}{4\pi} \int \left[ \vec{E}(\vec{r} \cdot \vec{E}) + \vec{B}(\vec{r} \cdot \vec{B}) - \vec{E}(\vec{r} \cdot \vec{B}) - \vec{B}(\vec{r} \cdot \vec{E}) \right]
\]

\[
\text{Now } \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) = \frac{1}{c^2} \left( \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right) + \frac{1}{c} \left( \vec{E} \times \frac{\partial \vec{B}}{\partial t} \right) - \frac{1}{c^2} \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) \quad \text{use } \vec{v} \times \vec{E} = -\frac{1}{c^2} \frac{\partial \vec{B}}{\partial t}
\]

\[
= \frac{1}{c^2} \left( \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right) - \vec{E} \times (\vec{v} \times \vec{E})
\]

\[
= -\frac{1}{c^2} \left( \frac{\partial \vec{E}}{\partial t} \times \vec{B} \right) + \vec{E} \times (\vec{v} \times \vec{E})
\]

Therefore,

\[
\vec{p}_E + \frac{1}{c^2} \vec{v} \times \vec{B} = \frac{1}{4\pi} \int \left[ \vec{E}(\vec{r} \cdot \vec{E}) + \vec{B}(\vec{r} \cdot \vec{B}) - \vec{E}(\vec{r} \cdot \vec{B}) - \vec{B}(\vec{r} \cdot \vec{E}) \right]
\]

Define electromagnetic momentum density

\[
\vec{\Pi} = \frac{1}{c^2} \vec{E} \times \vec{B}
\]

Then

\[
\frac{d \vec{P}_{\text{mech}}}{dt} + \frac{d}{dt} \int \vec{\Pi} = \frac{1}{4\pi} \int \left[ \vec{E}(\vec{r} \cdot \vec{E}) - \vec{E}(\vec{r} \cdot \vec{B}) + \vec{B}(\vec{r} \cdot \vec{B}) - \vec{B}(\vec{r} \cdot \vec{E}) \right]
\]

\[
\text{want to rewrite as a surface integral}
The component of integrand on right hand side is (E^2) part only
(sum over repeated indices)

\[ E_i \partial_j E_j - E_{ijk} E_j E_{kem} \partial_k E_m \]

\[ = E_i \partial_j E_j - (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) E_j \partial_k E_m \]

\[ = E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i \]

\[ = \partial_j \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) \]

Define Maxwell's stress tensor

\[ T_{ij} = \frac{1}{2} \left[ E_i E_j + B_i B_j - \frac{1}{2} \delta_{ij} (E^2 + B^2) \right] \]

(mote \( T_{ij} = T_{ji} \) Symmetric tensor)

Then

\[ \frac{d}{dt} \rho_{\text{mech}} + \frac{d}{dt} \int d^3r \, \Pi_i = \int d^3r \, \partial_j T_{ij} \]

\[ = \oint da \, T_{ij} \hat{n}_j \]

\[ \frac{d}{dt} \rho_{\text{mech}} + \frac{d}{dt} \int d^3r \, \Pi = \oint S \, da \, \frac{\partial}{\partial x_i} F_i \hat{N} \]

- \( T_{ij} \) gives the flow of the \( i^{th} \) component of electromagnetic field momentum through an element of surface area \( S \) to direction \( \hat{n}_j \)
Note: $\frac{d\mathbf{P}}{dx}$ is equal to the total electromagnetic force on the volume $V$.

Here we can write

$$\mathbf{F}_{\text{Em}} = \oint_S \mathbf{T} \cdot \hat{n} \, dA - \int_V \nabla \cdot \mathbf{B} \, dV$$

For static situations, the second term vanishes and

$$\mathbf{F}_{\text{Em}} = \oint_S \mathbf{T} \cdot \hat{n} \, dA$$

the $ij$ component of static force on unit area with normal $\hat{e}_j$.

This is origin of the term "stress" tensor.

$\mathbf{T}$ is like the stress tensor of an elastic medium.

$T_{xx}, T_{yy}, T_{zz}$ are like pressure.

Off-diagonal element are like shear stresses.
Force on a conductor surface.

Net force on surface per unit area is

\[ \mathbf{F} = \mathbf{\dot{E}}_{\text{above}} \cdot \mathbf{\hat{m}} - \mathbf{\dot{E}}_{\text{below}} \cdot \mathbf{\hat{m}} \]

\( t = 0 \) as \( \mathbf{\dot{E}} = 0 \) inside conductor

\[ \mathbf{F} = \frac{\mathbf{\dot{E}}}{4\pi} \left[ \mathbf{\hat{m}} \cdot (\mathbf{\ddot{E}} - \mathbf{\dot{E}}^2) \right] \]

For conductive surface

\[ \mathbf{\hat{m}} \cdot \mathbf{\ddot{E}}_{\text{above}} = \frac{4\pi}{\varepsilon_0} \delta \quad \text{(since} \mathbf{\dot{E}}_{\text{below}} = 0) \]

and tangential component \( \mathbf{\dot{E}} = 0 \)

\[ \Rightarrow \mathbf{\dot{E}} = \frac{4\pi}{\varepsilon_0} \mathbf{\hat{m}} \]

So

\[ \mathbf{\ddot{F}} = \frac{\mathbf{\dot{E}}}{4\pi} \left[ (\frac{4\pi}{\varepsilon_0} \mathbf{\hat{m}}) (\frac{4\pi}{\varepsilon_0}) - \frac{1}{2} \mathbf{\hat{m}} (\frac{4\pi}{\varepsilon_0})^2 \right] \]

\[ \mathbf{\ddot{F}} = \mathbf{\hat{m}} \left[ (\frac{4\pi}{\varepsilon_0})^2 - \frac{1}{2} (\frac{4\pi}{\varepsilon_0})^2 \right] = 2\pi \varepsilon_0^2 \mathbf{\hat{m}} \]

Force per unit area

\[ \mathbf{F} = 2\pi \varepsilon_0^2 \mathbf{\hat{m}} = \frac{1}{2} \varepsilon \mathbf{\dot{E}} \]

**Note:**

\( \frac{\mathbf{\dot{E}}}{\varepsilon_0} \).

Namely, one might have thought \( \mathbf{\ddot{F}} = \frac{\mathbf{\dot{E}}}{\varepsilon_0} \). But need

where \( \mathbf{\dot{E}}_{\text{ave}} = \frac{1}{2} (\mathbf{\dot{E}}_{\text{above}} + \mathbf{\dot{E}}_{\text{below}}) \) to exclude self-field of charge on

average field at surface

Surface from acting on itself, see

averaging over above + below

also Jackson pp 92 for another approach.
Consider a set of conductors with potential \( \phi(\vec{r}) = V_c \) fixed on conductor \( i \).

(Also need condition on \( \phi(\vec{r}) \to \infty \) if system is not enclosed)

From uniqueness theorem we know that specifying the \( V_c \) on each conductor is enough to determine the potential \( \phi(\vec{r}) \) everywhere. We can write this potential in the following form.

Let \( \phi^{(i)}(\vec{r}) \) be the solution to the boundary value problem

\[
\nabla^2 \phi^{(i)}(\vec{r}) = 0 \quad \text{and} \quad \phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } i, \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } j, \quad j \neq i \end{cases}
\]

Then by superposition

\[
\phi(\vec{r}) = \sum_i V_c \phi^{(i)}(\vec{r})
\]

is a solution to the problem \( \nabla^2 \phi = 0 \) and \( \phi(\vec{r}) = V_c \) for \( \vec{r} \) on surface of conductor \( i \).

The surface charge density at \( \vec{r} \) on surface of conductor \( i \) is

\[
\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi} \frac{\partial \phi^{(i)}(\vec{r})}{\partial \vec{m}} = -\frac{1}{4\pi} \frac{\partial}{\partial \vec{m}} \frac{\vec{r}}{\epsilon} \cdot \nabla \phi^{(i)}(\vec{r})
\]

where \( \frac{\partial \phi}{\partial \vec{m}} = \nabla \phi \cdot \hat{m} \) is the derivative normal to the surface at point \( \vec{r} \),
The total charge on conductor \( i \) is

\[
Q_i = \oint_{S_i} d\alpha \; \sigma_i(\alpha) = -\frac{1}{4\pi} \sum_j V_j \oint_{S_i} d\alpha \frac{\partial \Phi_j}{\partial m}
\]

\( \uparrow \) surface of conductor \( i \)

Define \( C_{ij} \equiv -\frac{1}{4\pi} \oint_{S_i} d\alpha \frac{\partial \Phi_j}{\partial m} \)

the \( C_{ij} \) depend only on the geometry of the conductors

Then we have

\[
Q_i = \sum_j C_{ij} V_j
\]

\( C_{ij} \) is the capacitance matrix

The charge on conductor \( i \) is a linear function of the potentials \( V_j \) on the conductors \( j \)

Since we know that specifying the \( Q_i \) that is on each conductor will uniquely determine \( \Phi(\vec{r}) \) and hence the potential \( V_i \) on each conductor, the capacitance matrix is invertible

\[
V_i = \sum_j \left[ C^{-1}\right]_{ij} Q_j
\]

The electrostatic energy of the conductors is then

\[
E = \frac{1}{2} \sum \left[ \frac{1}{2} \epsilon \sum_{i,j} C_{ij} V_i V_j \right]
\]
Compute to define capacitance of two conductors by

\[ C = \frac{Q}{V_1 - V_2} \]

when conductor 1 has charge \( Q \) and conductor 2 has charge \(-Q\). 

\( V_1 - V_2 \) is potential difference between the two conductors.

All other conductors fixed at \( V_0 = 0 \).

We can determine \( C \) in terms of the elements of the matrix \( C \):

\[ Q = C_{11}V_1 + C_{12}V_2 \]

\[ -Q = C_{21}V_1 + C_{22}V_2 \]

\[ \Rightarrow V_2 = -\frac{(C_{11} + C_{21})V_1}{C_{12} + C_{22}} \]

\[ V_1 - V_2 = \left[ 1 + \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right) \right] V_1 \]

\[ C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12} \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)}{1 + \left( \frac{C_{11} + C_{21}}{C_{12} + C_{22}} \right)} \]

\[ C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}} \]

\[ \]

Capacitance can also be defined when the space between the conductors is filled with a dielectric \( \varepsilon \). 
In this case, if \( q_0 \) is the free charge, then \( q_0 / \varepsilon \) is the effective total charge to use in computing \( \phi \).
\[ \frac{\Phi_i}{\varepsilon} = \sum_j C_{ij}^{(0)} V_j \]

where \( C_{ij}^{(0)} \) are capacitances appropriate to a vacuum between the conductors.

\[ A_i = \sum_j \varepsilon C_{ij}^{(0)} V_j \]

\[ = \sum_j C_{ij} V_j \quad \text{where} \quad C_{ij} = \varepsilon C_{ij}^{(0)} \]

the capacitance is increased by a factor the dielectric constant \( \varepsilon \).
Inductance

Consider a set of current carrying loops $C_i$ with currents $I_i$.

In Coulomb's gauge, we can write the magnetic vector potential $\vec{A}$ from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c} \int \frac{d^3r'}{\mid \vec{r} - \vec{r}' \mid} \vec{J}(\vec{r}') = \sum_i \frac{I_i}{c} \oint_{C_i} \vec{dl}' \cdot \frac{\vec{r} - \vec{r}'}{\mid \vec{r} - \vec{r}' \mid}$$

Integrate over loop $C_i$, integration variable is $\vec{r}'$.

The magnetic flux through loop $i$ is

$$\Phi_i = \oint_{S_i} \vec{m} \cdot \vec{B} = \oint_{S_i} \vec{m} \cdot \vec{n} \times \vec{A} = \oint_{C_i} \vec{dl} \cdot \vec{A}$$

Surface bounded by loop $C_i$.

$$\Phi_i = \oint_{C_i} \frac{\vec{I} \cdot \vec{r}}{c^2} \oint_{C_j} \frac{\vec{r}'}{\mid \vec{r} - \vec{r}' \mid}$$

Pure geometrical quantity.

$$\Phi_i = c \sum_j M_{ij} I_j$$

where $M_{ij} = \oint_{C_i} \oint_{C_j} \frac{\vec{dl}_i \cdot \vec{dl}_j}{c^2 \mid \vec{r} - \vec{r}' \mid}$

Is the mutual inductance of loops $i$ and $j$. $M_{ij} = M_{ji}$. 

\[ L_i = M_{ii} \text{ is self-inductance of loop (i)} \]

The sign convention in the above is that 
\[ \Phi_i \] is computed in direction given by right hand rule, according to the direction taken for current in loop (i)

\[ \Phi_i \]

Magnetic energy

\[ \mathcal{E} = \frac{1}{2c} \int d\mathbf{s} \cdot \mathbf{A} = \frac{1}{2c} \sum_i \oint \mathbf{d}l \cdot \mathbf{A} I_i \]

\[ = \frac{1}{2c} \sum_i \Phi_i I_i \]

\[ \mathcal{E} = \frac{1}{2} \sum_{i,j} M_{ij} I_i I_j \]