Force and torque on electric dipoles

Localized charge distribution $f(r)$ with net charge $\int d^3r \cdot f = 0$

Force on $f$ in slowly varying electric field $\vec{E}$ is

$$\vec{F} = \int d^3r \; f(r) \; \hat{E}(r)$$

define $\vec{r} = \vec{r}_0 + \vec{r}'$ where $\vec{r}_0$ is some fixed reference point in center of charge distribution $f$, and $\vec{r}'$ is distance relative to $\vec{r}_0$

$$\vec{F} = \int d^3r' \; f(\vec{r}') \; \hat{E}(\vec{r}_0 + \vec{r}')$$

Since $\vec{E}$ is slowly varying on length scale where $f \neq 0$, we expand

$$\vec{F} \approx \int d^3r' \; f(\vec{r}') \left[ \hat{E}(\vec{r}_0) + (\vec{r}' \cdot \nabla) \hat{E}(\vec{r}_0) \right] + \cdots$$

$$= \hat{E}(\vec{r}_0) \int d^3r' \; f(\vec{r}') + \left( \int d^3r' \; f(\vec{r}') \vec{r}' \cdot \nabla \right) \hat{E}(\vec{r}_0)$$

$$= 0 + (\vec{\phi}, \nabla) \hat{E}(\vec{r}_0)$$

$$\vec{F} = (\vec{\phi}, \nabla) \hat{E} = \sum_{\alpha} \vec{p}_\alpha \frac{\partial \hat{E}}{\partial \vec{q}_\alpha}$$

For $\vec{E} = \text{constant}$, $\vec{F} = 0$
Torque on a magnetic dipole

\[ \vec{N} = \int d^3r \, \vec{p}(\vec{r}) \times \vec{E}(\vec{r}) \approx \int d^3r \, \vec{p}(\vec{r}) \times [\vec{E}(\vec{r}) + \cdots] \]

to lowest order \[ \vec{N} = \vec{P} \times \vec{E} \]

Force and torque on magnetic dipoles

localized magnetostatic current distribution \( \vec{j}(\vec{r}) \)

\[ \vec{F} = \frac{1}{c} \int d^3r \, \vec{J} \times \vec{B} \]

expand about center of current \( \vec{r}_0 \)

\[ \vec{B}(\vec{r}) \approx \vec{B}(\vec{r}_0) + (\vec{r} - \vec{r}_0) \vec{b}(\vec{r}_0) + \cdots \]

\[ \vec{F} = \frac{1}{c} \left[ \int d^3r' \, \vec{j}(\vec{r}') \times \vec{b}(\vec{r}_0) + \frac{1}{c} \int d^3r' \, \vec{j}(\vec{r}') \times (\vec{r}' - \vec{r}_0) \vec{b}(\vec{r}_0) \right] \]

from discussion of magnetic dipole approx we had \( \int d^3r \vec{j} = 0 \) for magnetostatics, where \( \nabla \cdot \vec{j} = 0 \), so 1st term vanishes.

The 2nd term can be written as

\[ \vec{F}_d = \frac{\varepsilon_0 \mu_0}{c} \int d^3r' \, \vec{j}_d \, r'_s \times \vec{r}'_d \, \vec{B}_r \]

we need the tensor \( \frac{1}{4} \int d^3r' \, \vec{r}'_d \times \vec{r}'_s - \vec{r}'_d \times \vec{r}'_s \)

\[ = \frac{1}{2c} \int d^3r' \left[ \vec{r}'_d \times \vec{r}'_s - \vec{r}'_s \times \vec{r}'_d \right] \]

we get

\[ \vec{m} = \frac{1}{2} \varepsilon_0 \mu_0 c \int d^3r \, \vec{J} \times \vec{r} \]

\( \vec{m} \) magnetic dipole

\( m = \frac{1}{2} \varepsilon_0 \mu_0 c \int d^3r \, \vec{J} \times \vec{r} \)
\[ F_a = \varepsilon \partial \times E \times \partial (m \sigma) - \partial_x B_x \]
\[ = \left( \varepsilon \partial_x \partial_y \partial_z - \varepsilon \partial_y \partial_z \partial_x \right) m \sigma \partial_x B_x \]
\[ = \varepsilon \partial_m \left( \partial \times (m \cdot \vec{B}) \right) - \varepsilon \partial_x \partial \times \vec{B} \]
\[ \vec{F} = \vec{\nabla} \left( m \cdot \vec{B} \right) \quad \text{as} \quad \vec{F} \cdot \vec{B} = 0 \]

Torque on \( \vec{j} \):
\[ \vec{N} = \frac{1}{c} \int d^3r (\vec{r} \times (\vec{r} \times \vec{B})) \quad \text{to lowest order,} \quad \vec{B} = \vec{B}(\vec{r}) \]
\[ = \frac{1}{c} \int d^3r \left[ \vec{r} \times (\vec{r} \cdot \vec{B}) - \vec{B}(\vec{r} \cdot \vec{f}) \right] \]

2nd term \( = 0 \) as follows:
\[ \int d^3r \vec{r} \cdot \vec{f} = \int d^3r \vec{r} \cdot \vec{\nabla}(\vec{r}^2) \quad \text{as} \quad \vec{\nabla}(\vec{r}^2) = \vec{r} \]
\[ = -\int d^3r (\vec{r} \cdot \vec{f}) (\vec{r}^2) \quad \text{integrate by parts, surface term \( \to 0 \) as} \quad \vec{j} \quad \text{is localized} \]
\[ = 0 \quad \text{as} \quad \vec{\nabla} \cdot \vec{j} = 0 \quad \text{in magnetostatics} \]

1st term involves:
\[ \int d^3r \vec{j} \cdot \vec{r} = -\int d^3r \vec{r} \cdot \vec{j} = \frac{1}{2} \int d^3r \left[ \vec{F} \cdot \vec{r} - \vec{r} \cdot \vec{F} \right] \]

So,
\[ \vec{N} = \frac{1}{2c} \int d^3r \left[ \vec{r} \cdot \vec{F} (\vec{r} \cdot \vec{B}) - \vec{F} (\vec{r} \cdot \vec{B}) \right] \]
\[ \vec{N} = \frac{1}{2c} \int d^3r \left( \vec{r} \times \vec{B} \right) \]

\[ = \frac{1}{2c} \int d^3r \left( \vec{r} \times \vec{j} \right) \times \vec{B} \]

\[ \vec{N} = \vec{m} \times \vec{B} \]
Electrostatic energy of interaction

\[ E = \frac{1}{8\pi} \int d^3r \ E^2 \]

Suppose the charge density \( \rho \) that produces \( \vec{E} \) can be broken into two pieces, \( \rho = \rho_1 + \rho_2 \), with \( \vec{E} = \vec{E}_1 + \vec{E}_2 \) where \( \nabla \cdot \vec{E}_1 = 4\pi \rho_1 \) and \( \nabla \cdot \vec{E}_2 = 4\pi \rho_2 \).

Then

\[ E = \frac{1}{8\pi} \int d^3r \left[ (\vec{E}_1)^2 + (\vec{E}_2)^2 + 2 \vec{E}_1 \cdot \vec{E}_2 \right] \]

"self-energy" "self-energy" "interaction" energy

of \( \rho_1 \) of \( \rho_2 \) of \( \rho_1 \) with \( \rho_2 \)

\[ E_{\text{int}} = \frac{1}{4\pi} \int d^3r \vec{E}_1 \cdot \vec{E}_2 \]

\[ = \int d^3r \ \rho_1 \phi_2 = \int d^3r \ \rho_2 \phi_1 \]

where \( \vec{E}_1 = -\nabla \phi_1 \), \( \vec{E}_2 = -\nabla \phi_2 \), by similar manipulations as earlier.

Integrals are over all space.

Apply to the interaction energy of a dipole in an external \( \vec{E} \) field.

\[ E_{\text{int}} = \int d^3r \ \rho_1 \phi_2 \]

- potential of external \( \vec{E} \) field
- charge distribution of dipole
Assuming \( \phi \) varies on length scale of \( \rho \), then we can expand \( \phi_2(\vec{r}) = \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \nabla \phi_2(\vec{r}_0) \)

where \( \vec{r}_0 \) is the center of mass or any other convenient reference position within \( \rho \).

\[
\begin{align*}
E_{\text{int}} &= \int d^3r \, \rho_1(\vec{r}) \left[ \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \nabla \phi_2(\vec{r}_0) \right] \\
&= q \, \phi_2(\vec{r}_0) + \left[ \int d^3r \, \rho_1(\vec{r}) (\vec{r} - \vec{r}_0) \right] \cdot \nabla \phi_2(\vec{r}_0) \\
&= q \, \phi_2(\vec{r}_0) + \vec{\mu} \cdot \vec{E}
\end{align*}
\]

Where \( q \) is total charge in \( \rho_1 \) and \( \vec{\mu} \) is dipole moment with respect to \( \vec{r}_0 \). \( \vec{E} = -\nabla \phi_2 \) is external \( E \)-field.

For a neutral charge distribution \( q = 0 \) and \( \vec{\mu} \) is independent of the origin about which it is computed, so

\[
E_{\text{int}} = -\vec{\mu} \cdot \vec{E}
\]
< does not include the energy needed to make the dipole or to make \( \vec{E} \).

\( E_{\text{int}} \) is lowest when \( \vec{\mu} \parallel \vec{E} \).

\( \Rightarrow \) in thermal ensemble, dipoles tend to align parallel to an applied \( \vec{E} \).
Energy of magnetic dipole in external field

We had that the force on the dipole was

\[ \mathbf{F} = \nabla (m \cdot \mathbf{B}) \]

If we regard the force as coming from the gradient of a potential energy \( U \) then \( \mathbf{F} = -\nabla U \Rightarrow \)

\[ U = -m \cdot \mathbf{B} \]

or equivalently, energy = work done to move dipole into position from \( \mathbf{B} \)

\[ W = -\int_{0}^{F} \mathbf{F} \cdot d\mathbf{r} = -\int_{0}^{1} \nabla (m \cdot \mathbf{B}) \cdot d\mathbf{r} = \frac{m \cdot \mathbf{B}^2}{2} \]

This is the correct energy to use in cases where \( m \)

is due to intrinsic magnetic moments of atom or molecule—say from electron or nuclear spin. For a thermal ensemble, magnetic moments tend to align \( \parallel \) to \( \mathbf{B} \).

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we did if the the energy of an electric dipole in an electric field...
Magnetostatic energy of interaction

\[ E = \frac{1}{8\pi} \int d^3r \ B^2 \]

Suppose current \( \mathbf{j} \) that produces \( \mathbf{B} \) can be divided

\[ \mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 \quad \text{with} \quad \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2, \quad \text{where} \quad \mathbf{\nabla} \times \mathbf{B}_1 = \frac{\mu_0}{c^2} \mathbf{j}_1 \]

and \( \mathbf{\nabla} \times \mathbf{B}_2 = \frac{\mu_0}{c^2} \mathbf{j}_2 \). Then

\[ E = \frac{1}{8\pi} \int d^3r \left[ \mathbf{B}_1^2 + \mathbf{B}_2^2 + 2 \mathbf{B}_1 \cdot \mathbf{B}_2 \right] \]

- self energy
- self energy
- interaction energy
- \( \mathbf{B}_1 \)
- \( \mathbf{B}_2 \)
- of \( \mathbf{j}_1 \) with \( \mathbf{j}_2 \)

\[ E_{\text{int}} = \frac{1}{4\pi} \int d^3r \ \mathbf{B}_1 \cdot \mathbf{B}_2 \]

\[ = \frac{1}{c} \int d^3r \ \mathbf{j}_1 \cdot \mathbf{A}_2 = \frac{1}{c} \int d^3r \ \mathbf{j}_2 \cdot \mathbf{A}_1 \]

where \( \mathbf{B}_1 = \mathbf{\nabla} \times \mathbf{A}_1 \), \( \mathbf{B}_2 = \mathbf{\nabla} \times \mathbf{A}_2 \), by similar manipulations as earlier.

Integrals are over all space.

Apply to the interaction energy of a magnetic dipole in an external \( \mathbf{B} \) field.

\[ E_{\text{int}} = \frac{1}{2} \int d^3r \mathbf{j}_1 \cdot \mathbf{A}_2 \]

- \( \mathbf{A} \) vector potential of external \( \mathbf{B} \) field
- current distribution of dipole
Assume \( \hat{A} \) varies slowly on length scale of \( \ell \), then expand \( A_i(\mathbf{r}) = A_i(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla A_i(\mathbf{r}_0) \)

\[
E_{\text{int}} = \frac{1}{c} \int d^3r \quad \frac{1}{\epsilon} \cdot \hat{A}(\mathbf{r}_0)
+ \frac{1}{c} \int d^3r \quad \sum_{i,j} \frac{1}{\epsilon} x_{ij} (\mathbf{r} - \mathbf{r}_0) \cdot \frac{\partial}{\partial r_j} A_i(\mathbf{r}_0)
\]

From magnetostatic computation of magnetic dipole moment we had \( \int d^3r \hat{f} = 0 \) for magnetostatics

\( \Rightarrow 1^{\text{st}} \) term above vanishes. So does the piece of 2\text{nd} term \( (\int d^3r \hat{f}_{ij}) r_0 \cdot \frac{\partial}{\partial r_j} A_i(\mathbf{r}_0) \)

We are left with

\[
E_{\text{int}} = \left[ \frac{1}{c} \int d^3r \quad \frac{1}{\epsilon} x_{ij} \frac{\partial}{\partial r_j} A_i(\mathbf{r}_0) \right] \sum_{ij} \frac{\partial}{\partial r_j} A_i(\mathbf{r}_0)
\]

From computation of magnetic dipole approx
we had

\[
\int d^3r \hat{f}_{ij} = -\int d^3r \hat{x}_{ij} r_2
= \frac{1}{2} \int d^3r \left[ \hat{f}_{ij} r_2 - \hat{x}_{ij} r_2 \right]
= \frac{1}{2} \epsilon_{kij} \int d^3r \left( \hat{x}_k \times \hat{r} \right)_k
\]

Recall:

\[
\vec{m} = \frac{1}{2c} \int d^3r \quad \hat{x} \times \hat{r}
\]

\( \Rightarrow \frac{1}{2} \int d^3r \quad \hat{x}_{ij} r_2 = -\epsilon_{kij} m_k \quad \text{mag dipole moment} \)
\[ E_{\text{int}} = -m_0 \varepsilon_{kij} \partial_j A_i = m_0 \varepsilon_{kij} \partial_j A_i \]
\[ = \vec{m} \cdot (\nabla \times \vec{A}) \]
\[ = \vec{m} \cdot \vec{B} = E_{\text{int}} \]

This is opposite in sign to what we found earlier!

Why the difference?

1. When we integrate the work done against the magnetic force to move \( m \) into position from infinity, we found the energy \( U = -m \cdot \vec{B} \).

2. When we compute the interaction energy from
\[ E_{\text{int}} = \frac{1}{c^2} \int \frac{d^3r}{d^3r} \vec{r}_1 \cdot \vec{B}_2 \]
\[ = \frac{1}{c^2} \int \text{d}^3r \left( \frac{\vec{f}_1(\vec{r}) \cdot \vec{B}_2(\vec{r})}{|\vec{r} - \vec{r}_1|} \right) \]

we found the energy \( E_{\text{int}} = +m \cdot \vec{B} \).

To see which is correct, let us consider computing the interaction energy \( E_{\text{int}} \) directly via method 1.
Consider two loops with currents $I_1$ and $I_2$.

What is the work done to move loop 2 in from infinity to its final position with respect to loop 1?

Magnetostatic force on loop 2 due to loop 1 is

$$\vec{F} = \frac{I_2}{c} \oint_{\partial L_2} \vec{l} \times \vec{B}_1$$

Lorentz force

$$\vec{B}_1(r) = \frac{I_1}{c} \oint_{\partial L_1} \vec{l} \times \left( \vec{r} - \vec{r}_1 \right)$$

Biot-Savart law

$$F = \frac{I_1 I_2}{c^2} \oint_{\partial L_2} \oint_{\partial L_1} \vec{l} \times \left( \vec{l} \times \left( \vec{r}_2 - \vec{r}_1 \right) \right) \frac{1}{|\vec{r}_2 - \vec{r}_1|^3}$$

Use triple product rule

$$\vec{l} \times \left[ \vec{l} \times \left( \vec{r}_2 - \vec{r}_1 \right) \right] = \vec{l} \left[ \vec{l} \cdot \left( \vec{r}_2 - \vec{r}_1 \right) \right] - \left( \vec{r}_2 - \vec{r}_1 \right) \left( \vec{l} \cdot \vec{l} \right)$$

from the 1st term

$$\oint_{\partial L_2} \vec{l} \cdot \left( \vec{r}_2 - \vec{r}_1 \right) \frac{1}{|\vec{r}_2 - \vec{r}_1|^3} = -\oint_{\partial L_2} \vec{l} \cdot \hat{n} \frac{1}{|\vec{r}_2 - \vec{r}_1|^3} = 0$$

as integral of gradient around closed loop always vanishes!
\[ F' = -\frac{I_1 I_2}{C^2} \int \int d\mathbf{l}_1 \cdot d\mathbf{l}_2 \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|r_2 - r_1|^3} \]

Write \( \mathbf{r}_2 = \mathbf{R} + \delta \mathbf{r}_2 \) where \( \mathbf{R} \) is center of loop 2

Use \( \frac{\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|^3} = -\nabla_{\mathbf{R}} \left( \frac{1}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|} \right) \)

\[ \mathbf{F} = \frac{I_1 I_2}{C^2} \int \int d\mathbf{l}_1 \cdot d\mathbf{l}_2 \nabla_{\mathbf{R}} \left( \frac{1}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|} \right) \]

To move loop 2 we need to apply a force equal and opposite to the above magnetostatic force.

Therefore the work we do in moving loop 2 from infinity to its final position at \( \mathbf{R}_0 \) is

\[ W_{\text{mech}} = -\int_{\infty}^{\mathbf{R}_0} \mathbf{F} \cdot d\mathbf{R} = -\frac{I_1 I_2}{C^2} \int \int d\mathbf{l}_1 \cdot d\mathbf{l}_2 \int d\mathbf{R} \cdot \nabla_{\mathbf{R}} \left( \frac{1}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|} \right) \]

\[ = -\frac{I_1 I_2}{C^2} \int \int d\mathbf{l}_1 \cdot d\mathbf{l}_2 \left. \mathbf{R}_2 \right| \]

where \( \mathbf{r}_2 = \mathbf{R}_0 + \delta \mathbf{r}_2 \)

\[ = -\frac{1}{C^2} \int d^3r_1 \int d^3r_2 \left. \frac{f_1(r_1) \cdot f_2(r_2)}{|r_2 - r_1|} \right] \]

Note the minus sign! This is just the negative of the interaction energy!!

\[ = -M_{12} I_1 I_2 \]

\( \checkmark \) mutual inductance

Why the minus sign!
The minus sign we have here is the same
minus sign we get when we found \( U = -\vec{m} \cdot \vec{B} \)
by integrating the force on the magnetic dipole.

Why don't we get \(+\frac{1}{c^2} \int d^3r \, d^3r_2 \frac{\vec{f}_1(r) \cdot \vec{f}_2(r)}{r_{12}^2 - r_{12}}\)
with the plus sign we expect from \( E = \frac{1}{8\pi} \int d^3r \, B^2 \)?

Answer: we have left something out.

Faraday's law - when we move loop 2, the magnetic
flux through loop 2 changes. This \( \frac{d\Phi}{dt} \) creates
an emf \( \mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} \) around the loop that
would tend to change the current in the loop.

If we are to keep the current fixed at constant \( I_2 \)
then there must be a battery in the loop that does
work to counter this induced emf (electromotive force).

Similarly, the flux through loop 1 is changing and a
battery does work to keep \( I_1 \) constant. We need
to add this work done by the battery to the
mechanical work computed above.

emf induced in loop 1 \( \mathcal{E}_1 = \oint \mathbf{E}_1 \cdot d\mathbf{l} \) \[ \text{integrating in direction of current} \]
emf induced in loop 2 \( \mathcal{E}_2 = \oint \mathbf{E}_2 \cdot d\mathbf{l} \)

Faraday \( \mathcal{E}_1 = \frac{-d\Phi_1}{c dt} \) \( \mathcal{E}_1 = \text{flux through loop 1} \)

\( \mathcal{E}_2 = \frac{-d\Phi_2}{c dt} \) \( \mathcal{E}_2 = \text{flux through loop 2} \)
To keep the current constant, the batteries need to provide an emf that counters these Faraday-induced emfs. The work done by the battery per unit time is therefore

\[
\frac{dW_{\text{battery}}}{dt} = -\varepsilon_1 I_1 - \varepsilon_2 I_2
\]

(check units: \(\varepsilon I\) is \([\text{length}] \cdot [\text{force}] \cdot [1/\text{time}]\)

\[= [\text{length}] \cdot [\text{force}] \cdot [\text{energy}]\]

\[
\frac{dW_{\text{battery}}}{dt} = \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2
\]

\[
W_{\text{battery}} = \int_0^T dt \left( \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2 \right)
\]

where \(t = 0\) loop 2 is at infinity
\(t = T\) loop 2 is at final position
\(I_1\), \(I_2\) kept constant as loop moves

\[
W_{\text{battery}} = \frac{1}{2} \overline{\Phi}_1 I_1 + \frac{1}{2} \overline{\Phi}_2 I_2
\]

where \(\overline{\Phi}_1\) and \(\overline{\Phi}_2\)
are fluxes in final position, and are assumed that fluxes = 0 at infinity

\[
\overline{\Phi}_1 = CM_{12} I_2
\]

\[
\overline{\Phi}_2 = CM_{21} I_1 = CM_{12} I_1 \quad \text{as} \quad M_{12} = M_{21}
\]

\[\Rightarrow W_{\text{battery}} = 2M_{12} I_1 I_2\]
add this to the mechanical work

\[ W_{\text{total}} = W_{\text{mech}} + W_{\text{battery}} = -M_1 I_1 I_2 + 2M_2 I_1 I_2 \]

\[ = M_2 I_1 I_2 = \frac{1}{c^2} \int d^3r_1 \int d^3r_2 \frac{\mathbf{f}_1(r_1) \cdot \mathbf{f}_2(r_2)}{|r_1 - r_2|} \]

we get back the correct interaction energy!

Conclusion: The magnetostatic interaction energy

\[ \frac{1}{c^2} \int d^3r_1 \int d^3r_2 \frac{\mathbf{f}_1(r_1) \cdot \mathbf{f}_2(r_2)}{|r_1 - r_2|} \]

includes the work done to maintain the current stationery as the current distributions move.

When we computed the interaction energy of a current loop dipole \( \mathbf{m} \) and found

\[ W_{\text{int}} = +\mathbf{m} \cdot \mathbf{B} \]

this included the energy needed to maintain the constant current producing the constant \( \mathbf{m} \).

When we integrated the force on the dipole to find the potential energy

\[ U = -\mathbf{m} \cdot \mathbf{B} \]

this did not include the energy needed to maintain the constant current that creates \( \mathbf{m} \).

This is the correct energy expression to use when \( \mathbf{m} \) comes from intrinsic magnetic moments due to particles' intrinsic spin, which cannot be viewed as arising from a current loop!