Electromagnetic Waves in a vacuum

No sources $\vec{J}=0$, $\rho=0$

1) $\vec{\nabla}\cdot\vec{E}=0$ 3) $\vec{\nabla}\cdot\vec{B}=0$

2) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{c\,\partial t}$ 4) $\vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times(2) \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{E})}_{0 \text{ by } (1)} - \nabla^2\vec{E} = -\frac{\partial}{c\,\partial t}(\vec{\nabla}\times\vec{B})$$

$$-\nabla^2 E = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{c\,\partial t}\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Similarly

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

} wave equation
wave speed is c.

Note: in MKS units, above wave equation looks like

$$\nabla^2\vec{E} - \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

It was noticed that the speed of electromagnetic wave,

$\frac{1}{\sqrt{\varepsilon_0\mu_0}} = 3\times10^8$ m/s was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic wave

Harmonic

Plane waves

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$\vec{B}(\vec{r},t) = \mathrm{Re}\left[\vec{B}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

complex exponential form

$\vec{k}$ is wave vector

ω is angular frequency

$\nu = \omega/2\pi$ is frequency

$T = 1/\nu$ is period

$\lambda = \frac{2\pi}{|k|}$ is wavelength

$\left.\begin{array}{l}|\vec{E}_k| \\ |\vec{B}_k|\end{array}\right\}$ is amplitude

$\vec{E}(\vec{r}+\lambda\hat{k},t) = \vec{E}(\vec{r},t)$ periodic in space with period λ

$\vec{E}(\vec{r},t+T) = \vec{E}(r,t)$ periodic in time with period T

"plane wave" $\Rightarrow$ $\vec{E}(\vec{r},t)$ is constant in space on planes with normal $\hat{n} \parallel \vec{k}$.

properties of EM plane waves

$$\vec{\nabla}\cdot\vec{E} = 0 \implies \mathrm{Re}\left[\vec{E}_k\cdot\vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$= \mathrm{Re}\left[i\vec{E}_k\cdot\vec{k}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$
$$\Rightarrow \vec{E}_k\cdot\vec{k} = 0$$

amplitude is orthogonal to $\vec{k}$

$$\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \vec{B}_k\cdot\vec{k} = 0$$

amplitude orthogonal to $\vec{k}$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \mathrm{Re}\left[\vec{\nabla}\times\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \mathrm{Re}\left[\frac{1}{c}\vec{E}_k \frac{\partial}{\partial t} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \mathrm{Re}\left[-\vec{B}_k\times\vec{\nabla} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \mathrm{Re}\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \mathrm{Re}\left[i\vec{k}\times\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \mathrm{Re}\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \vec{k}\times\vec{B}_k = -\frac{\omega}{c}\vec{E}_k$$

$$\vec{k}\times\vec{k}\times\vec{B}_k = -k^2\vec{B}_k = -\frac{\omega}{c}\vec{k}\times\vec{E}_k$$

$$\underline{\vec{B}_k = \frac{\omega}{ck^2}\vec{k}\times\vec{E}_k}$$

<u>Finally</u>

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\Rightarrow \mathrm{Re}\left[\vec{E}_k\nabla^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \frac{\vec{E}_k}{c^2}\frac{\partial^2}{\partial t^2} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \mathrm{Re}\left[\vec{E}_k(-k^2)e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\omega^2}{c^2}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow k^2 = \frac{\omega^2}{c^2}$$

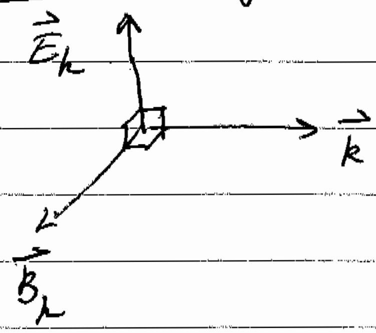
$$\boxed{\omega = \pm kc}$$ <u>dispersion relation</u>

insert in above

$$\vec{B}_k = \hat{k}\times\vec{E}_k \qquad \hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

$$\Rightarrow |\vec{B}_k| = |\vec{E}_k|$$

Summary



$\vec{E}_k \perp \vec{k}$
$\vec{B}_k \perp \vec{k}$
} "transverse" polarization

$\vec{B}_k = \hat{k} \times \vec{E}_k$

$\omega^2 = c^2 k^2$

$|\vec{B}_k| = |\vec{E}_k| \Rightarrow$ Lorentz force from plane EM wave on charge q is

$$q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

magnetic force is smaller factor $\left(\frac{v}{c}\right)$ as compared to electric force - can usually be ignored

Most general solution is a linear superposition of the above harmonic plane waves

$$\vec{E}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$ Fourier transform

$\vec{E}(\vec{r},t)$ is real $\Rightarrow$ $\vec{E}_k^{\,*} = \vec{E}_{-k}$

For dispersion relation $\omega^2 = c^2k^2$ we can write

$$\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot(\vec{r} - \vec{v}t)$$

where $\vec{v} = c\hat{k}$ is velocity of wave
If we only combine waves traveling in same direction $\hat{k}$, then

$$\vec{E}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \vec{E}_k \, e^{i\vec{k}\cdot(\vec{r} - \vec{v}t)} = \vec{E}(\vec{r} - \vec{v}t, 0)$$

the general plane wave solution of wave equation always has this property

$$\vec{E}(r,t) = \vec{E}(\vec{r} - \vec{v}t, 0)$$

If know $\vec{E}$ at $t=0$, then know $\vec{E}$ at all times t

Energy & momentum in EM wave

$$\left.\begin{aligned}\vec{E} &= Re\left[\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)\\ \vec{B} &= Re\left[\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \hat{k}\times\vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t)\end{aligned}\right\}\text{for real } \vec{E}_k$$

energy density $u = \frac{1}{8\pi}\left(E^2 + B^2\right)$

$$= \frac{1}{8\pi}\left[E_k^2 + E_k^2\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{1}{4\pi} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

Poynting vector

energy current

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$$

$$= \frac{c}{4\pi}\left[\vec{E}_k\times(\hat{k}\times\vec{E}_k)\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{c}{4\pi}\hat{k}\,E_k^2\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$\vec{S} = c\,u\,\hat{k}$$

momentum density $\vec{\Pi} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\hat{k}$

$u = c|\vec{\Pi}|$ — energy momentum relation of photons!

For visible light $\lambda \sim 5\times10^{-7}\,m \sim 5000\,\text{Å}$

$$T = \frac{\lambda}{c} = 1.6\times10^{-15}\,\text{sec}$$

most classical measurements on microscopic scales $t \gg T$, $\ell \gg \lambda$

measure average quantities

$$\langle u \rangle = \frac{1}{T}\int_0^T dt\ u = \frac{1}{8\pi} E_k^2 \qquad \text{as } \langle \cos^2\theta \rangle = 1/2$$

$$\langle \vec{S} \rangle = c \langle u \rangle \hat{k}$$

$$\langle \vec{\Pi} \rangle = \frac{1}{c} \langle u \rangle \hat{k}$$

intensity = average power per area transported by wave through surface with normal $\hat{n}$

$$I = \langle \vec{S} \rangle \cdot \hat{n}$$

Electromagnetic waves in matter

Macroscopic Maxwell equations with no sources
("free" charge and current vanishes)

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

linear materials

$$\vec{B} = \mu\vec{H}$$
$$\vec{D} = \epsilon\vec{E}$$

if μ and ϵ were simply constants then the above would become

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

Then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2 E = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$
$$= \frac{-1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial E}{\partial t}\right)$$
$$= -\frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$$

wave equation with wave speed $\frac{c}{\sqrt{\mu\epsilon}} < c$

This would be very much as for waves in a vacuum, except for the following minor changes:

$$\omega^2 = \frac{c^2 k^2}{\mu \epsilon}$$ dispersion relation changed by constant factor

$$\vec{E}_k \perp \vec{k}$$
$$\vec{B}_k \perp \vec{k}$$

$$i \vec{k} \times \vec{E}_k = \frac{i\omega}{c} \vec{B}_k$$

$$\frac{c|\vec{k}|}{\omega} \hat{k} \times \vec{E}_k = \vec{B}_k$$

$$\Rightarrow \sqrt{\mu\epsilon}\, \hat{k} \times \vec{E}_k = \vec{B}_k$$ $|\vec{B}_k| > |\vec{E}_k|$

wave speed $v = \frac{c}{\sqrt{\mu\epsilon}} < c$

In general however things are <u>much more complicated</u> because ϵ cannot be viewed as a constant when considering time varying behavior!