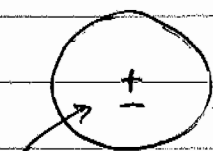


Time dependent polarizability of an atom



electron cloud

If displace center of electron cloud by a distance \vec{r} , there is a restoring

$$\text{force } \vec{F}_{\text{rest}} = -\frac{e^2 \vec{r}}{4\pi R^3} = -m\omega_0^2 \vec{r}$$

↑ ↑
electron mass resonant frequency

Also, in general there will be a damping force

$$\vec{F}_{\text{damp}} = -m\gamma \frac{d\vec{r}}{dt}$$

due to transfer of energy from atom to other degrees of freedom.

In an external electric field $\vec{E}(t)$, the equation of motion for electron cloud is

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}_{\text{tot}} = -e \vec{E}(t) - m\omega_0^2 \vec{r} - m\gamma \frac{d\vec{r}}{dt}$$

$$\ddot{\vec{r}} + \gamma \dot{\vec{r}} + \omega_0^2 \vec{r} = -\frac{e \vec{E}(t)}{m}$$

assuming \vec{E} is spatially constant over atomic distances

For harmonic oscillation $\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$

Assume solution $\vec{r}(t) = \vec{r}_0 e^{-i\omega t}$

(in the end, we will take the real parts)

Substitute into equation of motion

$$-\omega^2 \vec{r}_0 - i\omega \gamma \vec{r}_0 + \omega_0^2 \vec{r}_0 = -\frac{e \vec{E}_0}{m}$$

$$\vec{r}_0 = \frac{-e \vec{E}_0}{m(\omega_0^2 - \omega^2 - i\omega\gamma)}$$

polarization

$$\vec{p} = -e\vec{r} = \vec{p}_0 e^{-i\omega t}$$

$$\vec{p}_0 = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma)} \vec{E}_0 = \alpha(\omega) \vec{E}_0$$

$$\alpha(\omega) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \quad \text{freq dependent polarizability}$$

Since α is complex the polarization does not in general oscillate in phase with \vec{E} .

If $\alpha(\omega) = |\alpha| e^{i\delta}$ δ is phase of complex α

$$\vec{p}(t) = \alpha(\omega) \vec{E}(t) = |\alpha| e^{i\delta} \vec{E}_0 e^{-i\omega t} = |\alpha| \vec{E}_0 e^{-i(\omega t - \delta)}$$

↑
phase shifted by δ

For a general electric field

$$\vec{E}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega^* = \vec{E}_{-\omega}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} p_\omega e^{-i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) \vec{E}_\omega e^{-i\omega t}$$

Substitute in $\vec{E}_\omega = \int_{-\infty}^{\infty} dt' E(t') e^{i\omega t'}$ to get

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \alpha(\omega) e^{-i\omega(t-t')}$$

$$\vec{p}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \tilde{\alpha}(t-t')$$

↑ Fourier transf of $\alpha(\omega)$

\vec{p} at time t is due to \vec{E} at all times t'
non local in time

$\tilde{x}(t)$ is the response to $\vec{E}(t) = \delta(t)$

For our simple model

$$\tilde{x}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

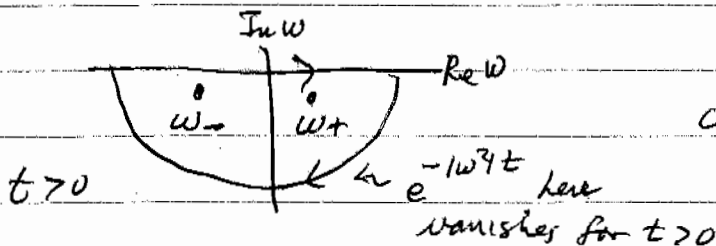
do by contour integration

$$\frac{1}{\omega^2 + i\gamma\omega - \omega_0^2} = \frac{1}{(\omega - \omega_+) (\omega - \omega_-)}$$

$$\omega_{\pm} = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = -\frac{i\gamma}{2} \pm \bar{\omega}$$

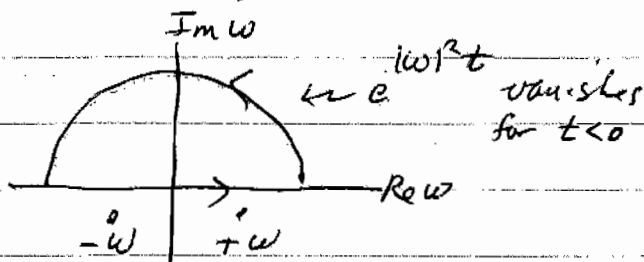
poles at ω_{\pm} are in lower half complex plane.

for $t > 0$, close contour in lower half plane



contour encloses poles
+ get contribution

for $t < 0$, close contour in upper half plane



contour encloses
no poles \Rightarrow integral
vanishes

$$\tilde{x}(t) = 0 \quad \text{for } t < 0$$

causal response! No polarization
until electric field turns on

For $t > 0$

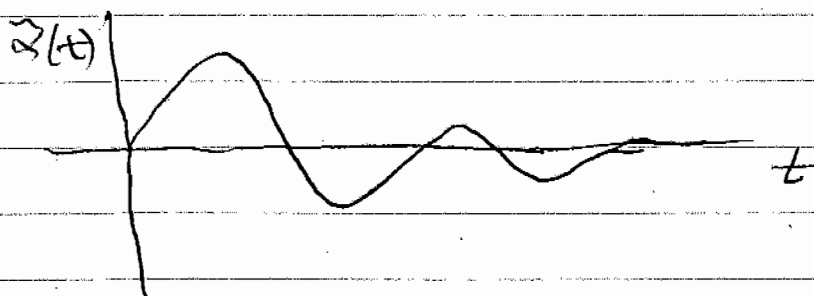
$$\tilde{x}(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \frac{e^2}{m} \frac{(-1)}{(\omega - \omega_+)(\omega - \omega_-)}$$

from Residue theorem

$$= (-2\pi i) \frac{e^2}{m} \frac{(-1)}{2\pi} \left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+} \right]$$

$$= \frac{ie^2}{m} \left[\frac{e^{-\gamma t/2} e^{-i\bar{\omega} t}}{2\bar{\omega}} - \frac{e^{-\gamma t/2} e^{i\bar{\omega} t}}{2\bar{\omega}} \right]$$

$$\tilde{x}(t) = \begin{cases} \frac{e^2}{m} \frac{e^{-\gamma t/2}}{\bar{\omega}} \sin(\bar{\omega} t) & t > 0 \\ 0 & t < 0 \end{cases}$$



damped oscillation

Polarization density $\vec{P}_\omega = 4\pi X(\omega) \vec{E}_\omega$ for harmonic oscillation

$X(\omega) \approx n \alpha(\omega)$ for dilute system

↑ atom density

can use Clausius-Mossotti correction for denser materials

$$\Rightarrow \vec{D}_\omega = \epsilon(\omega) \vec{E}_\omega \quad \epsilon(\omega) = 1 + 4\pi X(\omega)$$

↑ freq dependent

→ as with \vec{J} and \vec{E} , relation between \vec{D} and \vec{E} is non-local in time

$$\vec{D}(t) \neq \epsilon \vec{E}(t)$$

rather

$$\vec{D}(t) = \int_{-\infty}^{\infty} dt' \vec{E}(t') \tilde{\epsilon}(t-t')$$

↑ Fourier transf of $\epsilon(\omega)$

Ampere's law is

$$\vec{\nabla} \times \vec{H} = 4\pi \vec{J} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

becomes $\frac{1}{\mu} \vec{\nabla} \times \vec{B} = 4\pi \vec{J} + \frac{1}{c} \int_{-\infty}^{\infty} dt' \vec{E}(t') \frac{d}{dt} \tilde{\epsilon}(t-t')$

↪ integro-differential equation!

Maxwell's equations only look simple when expressed in terms of Fourier transforms

$$\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{B}(\vec{r}, t) = \vec{B}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{D}(\vec{r}, t) = \vec{D}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{H}(\vec{r}, t) = \vec{H}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Maxwell's Equ for source free system $\rho = \vec{J} = 0$

$$\vec{\nabla} \cdot \vec{D} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{c \partial t}, \quad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{c \partial t}$$

assume μ is true constant - not freq dependent
dielectric response is $\vec{D}_\omega = \epsilon(\omega) \vec{E}_\omega$

Then for the Fourier amplitudes of the fields, Maxwell's Equations become

transverse polarized

$$\begin{aligned} 1) \quad i \vec{k} \cdot \vec{D}_\omega &= i \epsilon(\omega) \vec{k} \cdot \vec{E}_\omega = 0 & \Rightarrow \boxed{\vec{k} \perp \vec{E}_\omega} & \text{(unless } \epsilon(\omega) = 0) \\ 2) \quad i \vec{k} \cdot \vec{B}_\omega &= 0 & \Rightarrow \boxed{\vec{k} \perp \vec{B}_\omega} \\ 3) \quad i \vec{k} \times \vec{E}_\omega &= i \frac{\omega}{c} \vec{B}_\omega \\ 4) \quad i \vec{k} \times \vec{H}_\omega &= -i \frac{\omega}{c} \vec{D}_\omega \Rightarrow \frac{i \vec{k}}{\mu} \times \vec{B}_\omega = -\frac{i \omega \epsilon(\omega)}{c} \vec{E}_\omega \end{aligned}$$

$$\begin{aligned} \vec{k} \times (3) &= i \vec{k} \times (\vec{k} \times \vec{E}_\omega) = i \frac{\omega}{c} \vec{k} \times \vec{B}_\omega \\ &\Rightarrow -i k^2 \vec{E}_\omega = -\frac{i \omega^2 \epsilon(\omega) \mu}{c^2} \vec{E}_\omega \quad \text{using (4)} \end{aligned}$$

$$\boxed{k^2 = \frac{\omega^2}{c^2} \epsilon(\omega) \mu} \quad \text{dispersion relation}$$

~~$$\vec{B}_\omega \times \vec{E}_\omega = \frac{c}{\omega} \vec{k} \times \vec{E}_\omega = \frac{c}{\omega} i \frac{\omega}{c} \vec{k} \times \vec{B}_\omega = i \vec{k} \times \vec{B}_\omega$$~~

Note: $\frac{\omega}{|\vec{k}|} = \frac{c}{\sqrt{\epsilon(\omega) \mu}}$

varies with ω .

there is not a single phase velocity.

$\Rightarrow \vec{E}$ is not in general a solution of a wave equation - different frequencies travel with different speeds

Since $\epsilon(\omega)$ is complex $\epsilon(\omega) = \epsilon_1(\omega) + i\epsilon_2(\omega)$

\Rightarrow wave vector also complex For $\vec{k} = k \hat{z}$

$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\epsilon_1 + i\epsilon_2}$$

$$\begin{aligned}\vec{E}(\vec{r}, t) &= \vec{E}_\omega e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_\omega e^{i[(k_1 + ik_2)z - \omega t]} \\ &= \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}\end{aligned}$$

k_1 determines the oscillation of the wave

k_2 determines the decay or attenuation of the wave as it propagates into the material

phase velocity $v_p \equiv \frac{\omega}{k_1}$

index of refraction $n = \frac{c}{v_p} = \frac{ck_1}{\omega}$

group velocity $v_g \quad \frac{1}{v_g} = \frac{dk_1}{d\omega}$

Magnetic field: $\vec{B}_\omega = \frac{c\vec{k}}{\omega} \times \vec{E}_\omega$

for $\vec{k} = k \hat{z}$, $\vec{B}_\omega = \frac{c(k_1 + ik_2)}{\omega} \hat{z} \times \vec{E}_\omega$

$$\begin{aligned}\text{if } k_1 + ik_2 &= \sqrt{k_1^2 + k_2^2} e^{i\delta} \quad \delta = \arctan\left(\frac{k_2}{k_1}\right) \\ &= |k| e^{i\delta}\end{aligned}$$

$$\vec{B}_\omega = \frac{c|k|}{\omega} \hat{z} \times \vec{E}_\omega e^{i\delta}$$

\uparrow phase shift

$$\vec{B}(\vec{r}, t) = \frac{c|k|}{\omega} (\hat{z} \times \vec{E}_\omega) e^{-k_2 z} e^{i(k_1 z - \omega t + \delta)}$$

Physical fields - take real parts

$$\vec{E}(\vec{r}, t) = \vec{E}_\omega e^{-k_2 z} \cos(k_1 z - \omega t)$$

$$\vec{B}(\vec{r}, t) = (\hat{z} \times \vec{E}_\omega) \frac{c|k|}{\omega} e^{-k_2 z} \cos(k_1 z - \omega t + \delta)$$

Conclusions

1) \vec{E} and $\vec{B} \perp \vec{k}$ transverse polarized

2) $\vec{E} \perp \vec{B}$

3) amplitude ratio $\frac{|\vec{B}|}{|\vec{E}|} = \frac{c|k|}{\omega} = \sqrt{|\epsilon(\omega)| \mu}$

4) \vec{B} is shifted in phase with respect to \vec{E} by phase shift $\delta = \arctan(k_2/k_1)$

5) waves decay as they propagate $e^{-k_2 z}$ } consequence of complex $\epsilon(\omega)$

If $\epsilon_2 = 0$, i.e. $\epsilon(\omega)$ is real, and if $\epsilon > 0$, then $k_2 = 0 \Rightarrow$ no decay, no phase shift

consequences of frequency dependence of $\epsilon(\omega)$

6) $\vec{E}(t)$ and $\vec{D}(t)$ non locally related in time

7) waves of different ω travel with different $v_p = \omega/k$

8) dispersion - wave pulses do not travel with v_p

and they spread as they propagate pulses travel with group velocity $v_g = \frac{d\omega}{dk}$ (see Quantum Mechanics discussion)

$v_g < v_p$ "normal dispersion"

$v_g > v_p$ "anomalous dispersion"

$$\frac{1}{v_g} = \frac{dk_1}{d\omega} = \frac{d}{d\omega} \left[\frac{\omega}{c} n \right]$$

index of refraction

$$\frac{1}{v_g} = \frac{n}{c} + \frac{\omega}{c} \frac{dn}{d\omega} = \frac{1}{v_p} + \frac{\omega}{c} \frac{dn}{d\omega}$$

$$v_g = \frac{v_p}{1 + \frac{v_p \omega}{c} \frac{dn}{d\omega}}$$

⇒ when $\frac{dn}{d\omega} > 0$, $v_g < v_p$ normal dispersion
 $\frac{dn}{d\omega} < 0$, $v_g > v_p$ anomalous dispersion