

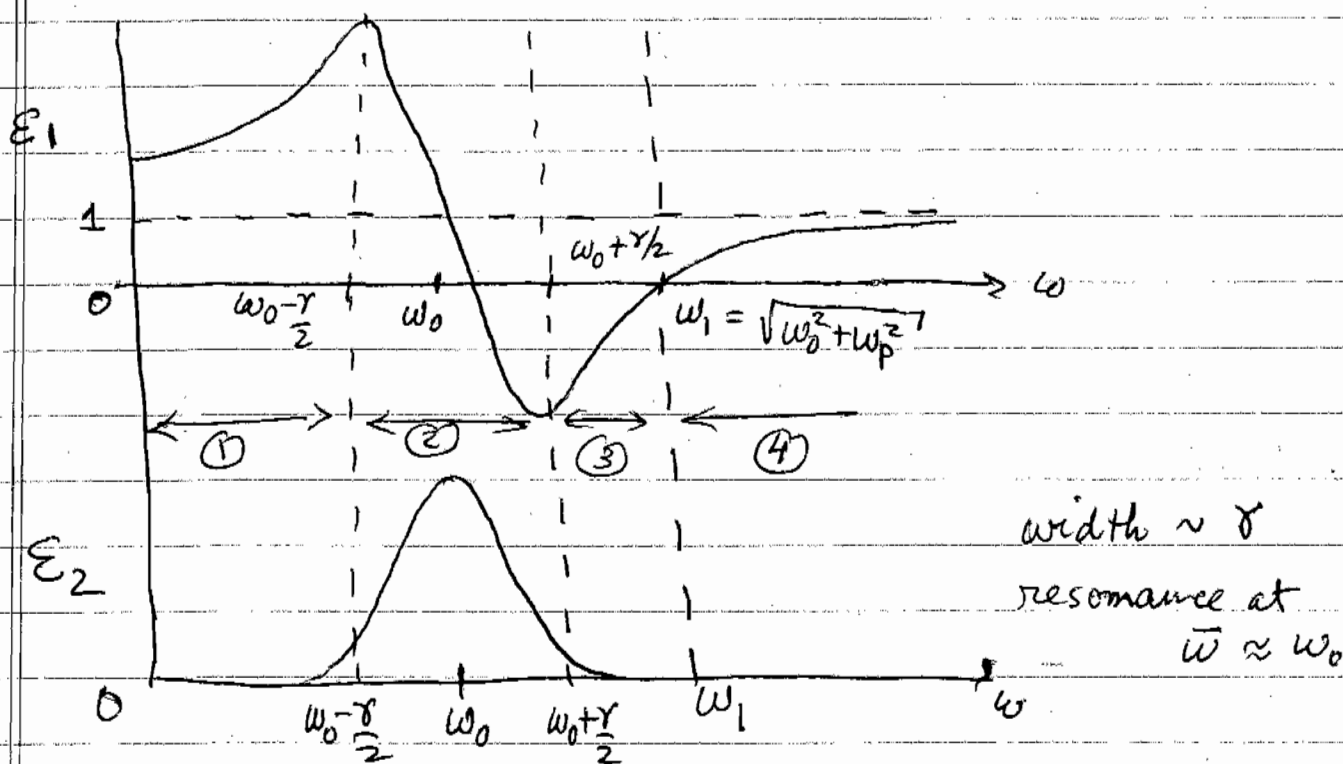
For our simple model: $\epsilon = 1 + 4\pi\chi \approx 1 + 4\pi\mu\alpha$

$$\epsilon(\omega) = 1 + \frac{4\pi m e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\epsilon_1 = 1 + \frac{4\pi m e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\epsilon_2 = \frac{4\pi m e^2}{m} \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

Define $\omega_p = \sqrt{\frac{4\pi m e^2}{m}}$ the "plasma frequency"



$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\epsilon_1 + i\epsilon_2}$$

$$k^2 = k_1^2 - k_2^2 + 2ik_1k_2 = \frac{\omega^2}{c^2} \mu (\epsilon_1 + i\epsilon_2)$$

equating real and imaginary pieces and
solve for k_1 and k_2

$$k_1 = \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \sqrt{\epsilon_1^2 + \epsilon_2^2} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \sqrt{\epsilon_1^2 + \epsilon_2^2} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$

Regions of different behavior

Regions ① and ④ - transparent propagation

$$\epsilon_1 > 0, \quad \epsilon_1 \gg \epsilon_2$$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_1 \left(1 + \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_1} \right)^2 \right) + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\epsilon_1 + \frac{1}{4} \frac{\epsilon_2^2}{\epsilon_1} \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu \epsilon_1} + \text{small correction}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_1 \left(1 + \frac{1}{2} \left(\frac{\epsilon_2}{\epsilon_1} \right)^2 \right) - \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{4} \frac{\epsilon_2^2}{\epsilon_1} \right]^{1/2} = k_1 \left(\frac{\epsilon_2}{2\epsilon_1} \right) \ll k_1$$

So $k_2 \ll k_1$ small attenuation

\Rightarrow medium is transparent

Note: $v_p = \frac{\omega}{k_1} = \frac{c}{n} = \frac{c}{\sqrt{\epsilon_1 \mu}}$

in region ①, $\epsilon_1 > 1 \Rightarrow v_p < c$

in region ②, $\epsilon_1 < 1 \Rightarrow v_p > c!$

but $v_g < c$ always!

Region ② $\omega \approx \omega_0$ resonant absorption

$$\epsilon_2 \approx \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{1}{\gamma}\right) \gg 1 \quad \text{for a sharp resonance with } \gamma \ll \omega_0$$
$$\epsilon_1 \approx 1$$

So $\epsilon_2 \gg \epsilon_1$

$$k_1 \approx \pm \frac{\omega \sqrt{\mu}}{c} \left[\frac{1}{2} \epsilon_2 \left(1 + \frac{1}{2} \left(\frac{\epsilon_1}{\epsilon_2} \right)^2 \right) + \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 + \frac{1}{4} \frac{\epsilon_1^2}{\epsilon_2} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\frac{1}{2} \epsilon_2}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 \left(1 + \frac{1}{2} \left(\frac{\epsilon_1}{\epsilon_2} \right)^2 \right) - \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} \epsilon_2 + \frac{1}{4} \frac{\epsilon_1^2}{\epsilon_2} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$
$$\approx \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\frac{1}{2} \epsilon_2}$$

$k_1 \approx k_2$ strong attenuation

wave excites atoms at resonance \Rightarrow large atomic displacements \rightarrow media absorbs most energy from the wave \Rightarrow wave decays rapidly, decreases factor $\frac{1}{e^{2\pi}}$ within one wavelength of propagation.

Region ③ $\epsilon_1 < 0$, $|\epsilon_1| \gg \epsilon_2$
total reflection

width of region ③ is

$$\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$$

increases with atomic density as $\omega_p \gg \omega_0$

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} |\epsilon_1| + \frac{1}{4} \frac{\epsilon_2^2}{|\epsilon_1|} + \frac{1}{2} \epsilon_1 \right]^{1/2}$$

↑ cancel as $|\epsilon_1| = -\epsilon_1$ ↓

$$k_1 \approx \pm \frac{\omega}{c} \sqrt{\mu |\epsilon_1|} \frac{\epsilon_2}{2|\epsilon_1|}$$

$$k_2 \approx \pm \frac{\omega}{c} \sqrt{\mu} \left[\frac{1}{2} |\epsilon_1| + \frac{1}{4} \frac{\epsilon_2^2}{|\epsilon_1|} - \frac{1}{2} \epsilon_1 \right]^{1/2}$$

$$\approx \pm \frac{\omega}{c} \sqrt{\mu |\epsilon_1|}$$

$$\frac{k_2}{k_1} = \frac{2|\epsilon_1|}{\epsilon_2} \gg 1$$

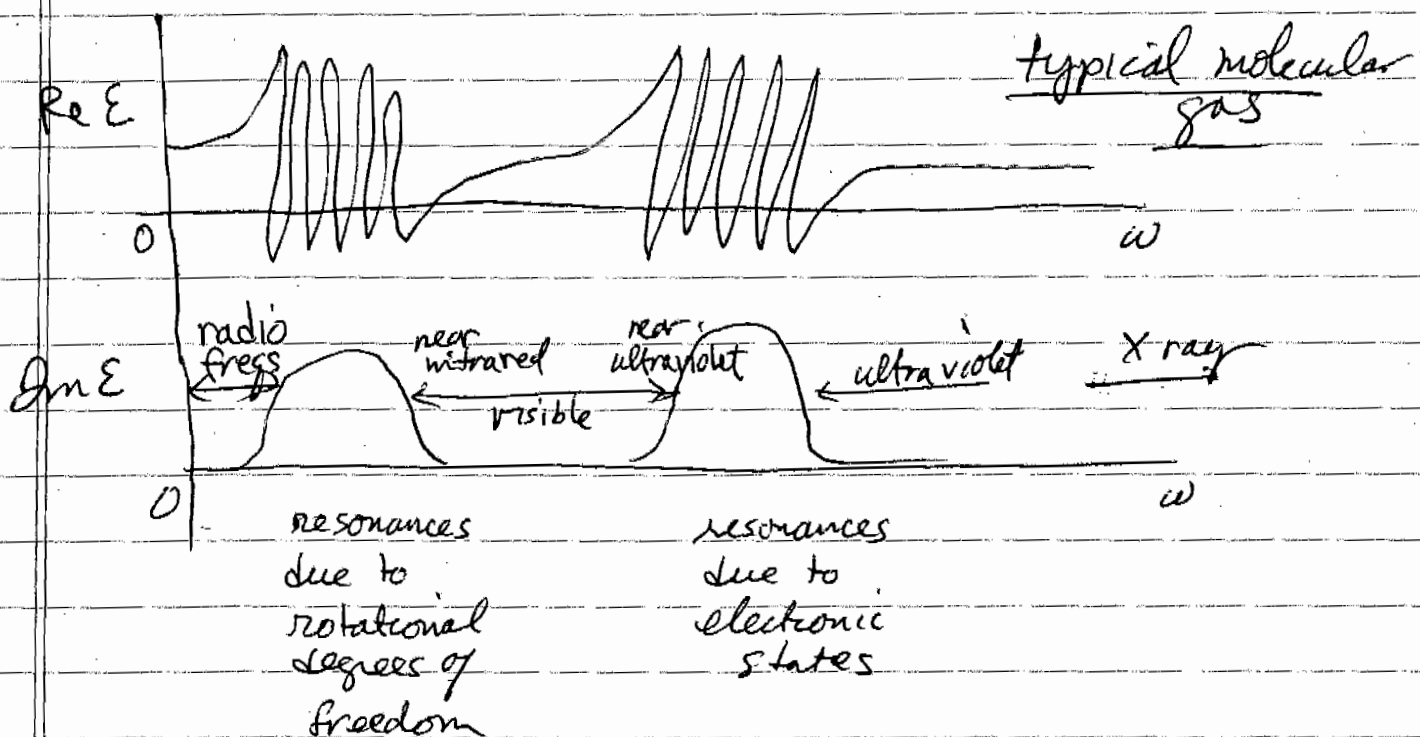
wave vector is almost pure imaginary
 wave decays exponentially to zero in much less than one wavelength.

we will see this corresponds to total reflection
 since $\omega \gg \omega_0$, we are not at resonance
 so material is not absorbing much energy from wave. The strong attenuation is due to the destructive interference between the wave and the induced fields of the polarized atoms

Our single model had a single resonance at ω_0 .
 A more realistic model for molecules has many bands of resonances due to rotational, vibrational, and electronic modes of excitation.

$$\epsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where f_i are spacings between energy levels with allowed electric dipole transitions



$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

$$= 4.4 \times 10^{16} \sqrt{\frac{m}{M_A}} \text{ sec}^{-1}, \quad M_A = 6 \times 10^{23} / \text{cm}^3$$

~~For H₂O~~

$$\Rightarrow \hbar \omega_p = 185 \sqrt{\frac{m}{M_A}} \text{ eV}$$

For H₂O $\frac{m}{M_A} \sim 0.05$

$$\hbar \omega_p \sim 40 \text{ eV}$$

For typical metal $\frac{m}{M_A} \sim 0.1$

$$\hbar \omega_p \sim 58 \text{ eV}$$

compared to $\hbar \omega_p \sim \text{eV}$

EM waves in Conductors

conduction electrons are mobile, not bound
⇒ we have to include the \vec{j}_f and ρ_f from them.

Simple classical model for electron motion - "Drude" Model

$$m \ddot{\vec{r}} = -e \vec{E}(t) - \frac{m}{\tau} \dot{\vec{r}}$$

↑ external E field ↑ damping force due to collisions
τ is "relaxation time"

$$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t} \text{ solution}$$

plug in to get

$$(-\omega^2 - \frac{i\omega}{\tau}) \vec{r}_\omega = -\frac{e}{m} \vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m} \frac{1}{\omega^2 + \frac{i\omega}{\tau}} \vec{E}_\omega$$

$$\text{current is } \vec{j}_f = -en \dot{\vec{r}}_\omega = -en(-i\omega) \vec{r}_\omega$$

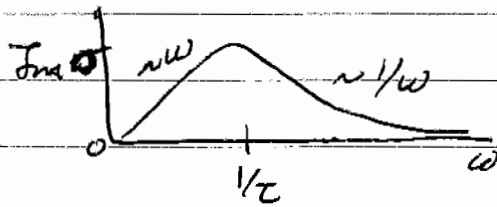
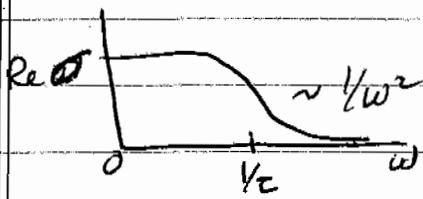
↑ density of electrons

$$\vec{j}_f = \frac{ne^2}{m} \frac{i\omega}{\omega^2 + \frac{i\omega}{\tau}} \vec{E}_\omega = \frac{me^2c}{m} \frac{1}{1 - i\omega\tau} \vec{E}_\omega$$

$$\vec{j}_f = \sigma(\omega) \vec{E}_\omega$$

conductivity

$$\sigma(\omega) = \frac{me^2c}{m} \frac{1}{1 - i\omega\tau}$$



$$\text{Re } \sigma = \frac{\sigma_0}{1 + \omega^2 \tau^2}$$

$$\text{Im } \sigma = \frac{\sigma_0 \omega \tau}{1 + \omega^2 \tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{m e^2 \tau}{m}$$

dc conductivity

Charge density ρ_f given by charge conservation law,
for plane waves

$$\rho_f = \rho_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad \vec{j}_f = \vec{j}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{j}_f \Rightarrow -i\omega \rho_0 = -i\vec{k} \cdot \vec{j}_0$$

$$\rho_0 = \frac{\vec{k} \cdot \vec{j}_0}{\omega} = \frac{\sigma(\omega) \vec{k} \cdot \vec{E}_\omega}{\omega}$$

Maxwell Equations

$$1) \quad \vec{\nabla} \cdot \vec{D} = 4\pi \rho_f$$

$$2) \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$3) \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$4) \quad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j}_f + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

Assume $\vec{H} = \vec{B}/\mu$, μ constant

$$\vec{D}_\omega = \epsilon_b(\omega) \vec{E}_\omega$$

$\epsilon_b(\omega)$ is dielectric function

$$\vec{j}_\omega = \sigma(\omega) \vec{E}_\omega$$

from the bound charges

$$\rho_\omega = \frac{\sigma}{\omega} \vec{k} \cdot \vec{E}_\omega$$

$\sigma(\omega)$ is conductivity from free charges

For harmonic plane wave solutions $\vec{E} = E_{\omega} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$
etc.

$$1) \Rightarrow i\vec{k}\cdot\vec{D}_{\omega} = i\vec{k}\cdot\epsilon_b E_{\omega} = 4\pi j_{\omega} = 4\pi\sigma \frac{\vec{k}\cdot\vec{E}_{\omega}}{\omega}$$

$$\Rightarrow i\vec{k}\cdot\vec{E}_{\omega} \left(\epsilon_b + \frac{4\pi i\sigma}{\omega} \right) = 0$$

$$2) \Rightarrow i\mu \vec{k}\cdot\vec{H}_{\omega} = 0$$

$$3) \Rightarrow i\vec{k}\times\vec{E}_{\omega} = \frac{i\omega}{c}\vec{B}_{\omega} = \frac{i\omega\mu}{c}\vec{H}_{\omega}$$

$$4) \Rightarrow i\vec{k}\times\vec{H}_{\omega} = \frac{4\pi}{c}\vec{j}_{\omega} - \frac{i\omega}{c}\vec{D}_{\omega}$$
$$= \frac{4\pi\sigma}{c}\vec{E}_{\omega} - \frac{i\omega}{c}\epsilon_b\vec{E}_{\omega}$$
$$= -\frac{i\omega}{c} \left(\epsilon_b + \frac{4\pi i\sigma}{\omega} \right) \vec{E}_{\omega}$$

Notice: all the equations above look exactly like what we had for the dielectric, provided we define

$$\epsilon(\omega) = \epsilon_b(\omega) + \frac{4\pi i\sigma(\omega)}{\omega}$$

So all results for the dielectric case carry over to conductors, provided we make the above substitution. In particular

dispersion relation for transverse modes $\left[k^2 = \frac{\omega^2}{c^2} \mu \epsilon(\omega) \right]$