The main difference between dielectrics and conductors has to do with the contribution that the \( 4\pi i\sigma /\omega \) makes to the real and imaginary parts of \( \varepsilon(\omega) \).

For single Drude model \( \sigma(\omega) = \frac{\sigma_0}{1-i\omega \tau} \quad \sigma_0 = \frac{me^2}{m} \)

1. Low frequencies \( \omega \ll \frac{1}{\tau} \)
   \[ \varepsilon_\infty(\omega) \approx \varepsilon_\infty(0) \text{ real} \]
   \[ \sigma(\omega) \approx \sigma_0 \text{ real} \]
   \[ \Rightarrow \varepsilon(\omega) \approx \varepsilon_\infty(0) + \frac{4\pi i\sigma_0}{\omega} \] (gives large \( \varepsilon_2 \) as \( \omega \to 0 \))
   
   \( \Re \varepsilon = \varepsilon_1 \)
   \( \Im \varepsilon = \varepsilon_2 \)

   when \( \frac{\varepsilon_2}{\varepsilon_1} \gg 1 \) we call this region a "good" conductor.

   - conduction electrons dominate the response
   - waves strongly attenuated

   when \( \frac{\varepsilon_2}{\varepsilon_1} \ll 1 \) we call this region a "poor" conductor.

   - little absorption of energy by conduction electrons

   waves propagate

   one always enters the "good" conductor region when \( \omega \) gets sufficiently small.
wave vector:

\[ k = \frac{\omega}{c} \sqrt{\mu \varepsilon} \]

for a good conductor where \( \varepsilon_2 \gg \varepsilon_1 \),

\[ \varepsilon \sim i \varepsilon_2 = \frac{4\pi i \sigma_0}{\omega} \]

\[ k = k_1 + i k_2 = \frac{\omega}{c} \sqrt{\mu \frac{4\pi i \sigma_0}{\omega}} \quad \sqrt{\varepsilon} = \frac{1 + i}{\sqrt{2}} \]

\[ k_1 = k_2 = \frac{\omega}{c} \sqrt{\frac{4\pi \mu \sigma_0}{2\omega}} = \frac{1}{c} \sqrt{2\pi \mu \sigma_0 \omega} \]

For \( k^2 = k_3^2 \),

\[ E = E_0 e^{i(k_3 - \omega t)} = E_0 e^{-k_2 z} e^{i(k_3 - \omega t)} \]

\[ s = \frac{1}{k_2} = \frac{c}{\sqrt{2\pi \mu \sigma_0 \omega}} \]

"Skin depth" 

\[ s \sim \frac{1}{\sqrt{\omega}} \text{ increases as } \omega \text{ decreases} \]

Phase shift between oscillations of \( \vec{E} \) and \( \vec{H} \)

\[ \phi = \arctan \left( \frac{k_2}{k_1} \right) \approx \arctan (1) = 45^\circ \]

Amplitude ratio:

\[ \frac{|\vec{H}|}{|\vec{E}|} = \frac{c(k_1)}{\omega \mu} = \frac{\sqrt{2} c}{\omega \mu} \]

\[ = \frac{\sqrt{2} c}{\omega \mu} \frac{1}{c} \sqrt{2\pi \mu \sigma_0 \omega} \]

\[ = \sqrt{\frac{4\pi \sigma_0}{\omega \mu}} \sim \frac{1}{\sqrt{\omega}} \]

as \( \omega \to 0 \), most of the energy of the wave is carried by the magnetic field part.
2. High frequencies: \( \omega \gg \frac{1}{\tau}, \omega \gg \omega_0 \)

\[
\varepsilon_b(\omega) \approx 1
\]

\[
\sigma(\omega) \approx \frac{\sigma_0}{-i \omega \tau} = \frac{i m e^2}{\omega \tau} = \frac{i m e^2}{\omega m}
\]

\[
\varepsilon(\omega) \approx 1 + \frac{4\pi i \sigma}{\omega} = 1 - \frac{4\pi m e^2}{\omega m^2}
\]

\[
\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}
\]

\[
\omega_p = \sqrt{\frac{4\pi m e^2}{\mu}}
\]

\(\varepsilon(\omega)\) is real

1) If \( \omega > \omega_p \) then \( \varepsilon > 0 \)

\( \Rightarrow \) transparent propagation

\[
k = k_1 = \frac{\omega}{c} \sqrt{\mu \varepsilon}
\]

\( k_2 \approx 0 \)

2) If \( \omega < \omega_p \) then \( \varepsilon < 0 \)

\( \Rightarrow \) total reflection

\[
k_1 \approx 0
\]

\[
k = k_2 = \frac{\omega}{c} \sqrt{\mu \varepsilon(\omega)}
\]

\( \omega_p \) gives crossover between total reflection and transparent propagation.
for typical metals

\[ T = 10^{-14} \text{ sec} \]

\[ \omega_p = 10^{16} \text{ sec}^{-1} \]

\[ 2p = \frac{2\pi c}{\omega_p} \sim 3 \times 10^3 \AA \] (visible \ i
\[ a \sim 5 \times 10^3 \AA \])

**Example:** The ionosphere is a layer of charged gas surrounding the Earth. In many respects, the charged particles of the ionosphere behave like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio \( W_{AM} < \omega_p \Rightarrow \) AM radio signals reflected back to Earth

for FM radio \( W_{FM} > \omega_p \Rightarrow \) FM radio signals propagate through ionosphere into space

Explain why you can pick up AM stations from far away - they get reflected back. But you can only pick up local FM stations.
Longitudinal modes in conductors

\[ \mathbf{F}_\text{m} = \mathbf{E}_\text{m} \times \mathbf{H}_\text{m} \]

Magnetic field

\[ \nabla \cdot \mathbf{B} = 0 \Rightarrow \mu \nabla \cdot \mathbf{H} = 0 \Rightarrow \mathbf{H} \perp \mathbf{k} \text{ transverse} \]

or \( k = 0 \) spatially uniform \( \mathbf{H} \)

\[ \varepsilon \mathbf{E}_\text{m} = \mathbf{E}_\text{m} \times \mathbf{H}_\text{m} = 0 \Rightarrow \omega = 0 \]

0° as \( \mathbf{E}_\text{m} = 0 \)

So only possible longitudinal \( \mathbf{H} \) is spatially uniform, constant in time.

Electric field

\[ \nabla \cdot \mathbf{D} = \rho \Rightarrow \varepsilon(\omega) \nabla \cdot \mathbf{E}_\text{m} = 0 \Rightarrow \mathbf{E}_\text{m} \perp \mathbf{k} \text{ transverse} \]

If \( \mathbf{E}_\text{m} \parallel \mathbf{k} \) but \( \varepsilon(\omega) = 0 \), then can satisfy all other Maxwell equations.

\[ i \mathbf{k} \times \mathbf{E}_\text{m} = i \omega \varepsilon(\omega) \mathbf{H}_\text{m} \Rightarrow \mathbf{H}_\text{m} = 0 \]

\[ \Rightarrow i \mu \mathbf{k} \cdot \mathbf{H}_\text{m} = 0 \quad \text{and} \quad \mathbf{k} \times \mathbf{H}_\text{m} = -\frac{i \omega \varepsilon(\omega)}{\varepsilon(\omega)} \mathbf{E}_\text{m} \]

0° as \( \mathbf{H}_\text{m} = 0 \) as \( \varepsilon(\omega) = 0 \)

So we can have longitudinal electric field oscillations when \( \varepsilon(\omega) = 0 \)
**Low freq:** \( \omega \ll \omega_0 \), \( \omega \approx 1 \)

\[ E \approx E(0) + \frac{4\pi i \sigma_0}{\omega} \]

\[ E(\omega) = 0 \text{ when } \omega = -\frac{4\pi i \sigma_0}{E(0)} \]

\[ \vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(k\cdot\vec{r} - \omega t)} = \vec{E}_0 e^{-i\frac{4\pi i \sigma_0}{E(0)} t} e^{-i\vec{k} \cdot \vec{r}} \]

If set up a longitudinal \( \vec{E} \) field, it decays to zero exponentially with temporal decay time \( E_0(0)/4\pi \sigma_0 \).

This is consistent with assumption the \( \vec{E} = 0 \) inside a conductor for electrostatics.

In statics \( \vec{E} = -\vec{\nabla} \phi \Rightarrow \vec{E} \rightarrow -i\vec{k} \vec{\phi}_k e^{-i\vec{k} \cdot \vec{r}} \) is longitudinal.

**High freq:** \( \omega \gg \gamma_c \), \( \omega \gg \omega_0 \)

\[ E(\omega) \approx 1 + \frac{4\pi i \sigma_0}{\omega_0} = 1 - \frac{\omega_0^2}{\omega^2} \quad \alpha_0^2 = \frac{4\pi \gamma_m e^2}{m} \]

\[ E = 0 \text{ when } \omega = \omega_0 \]

So we have oscillatory longitudinal \( \vec{E} \) only when \( \omega = \omega_0 \), independent of \( \vec{k} \).

\[ \vec{E} = \vec{E}_0 e^{-i\vec{k} \cdot \vec{r}} e^{-i\omega_c t} \]

This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a plasma oscillation.

\[ \nabla \cdot \vec{E} = 4\pi \sigma \Rightarrow \sigma = \frac{i\hbar \vec{E}_0}{4\pi} e^{-i\vec{k} \cdot \vec{r}} e^{-i\omega_c t} \]

\[ \int \text{plasma osc. } \]

\[ \text{a charge density oscillation} \]
Consider a transverse plane wave traveling in direction \( \hat{M} \), i.e. \( \vec{k} = k \hat{M} \). Define a right-handed coordinate system as follows:

\[
\begin{align*}
\hat{e}_1 &\times \hat{e}_2 = \hat{M} \\
\hat{M} \times \hat{e}_1 = \hat{e}_2 \\
\hat{e}_2 \times \hat{M} = \hat{e}_1
\end{align*}
\]

A general solution to Maxwell's equations for a transverse plane wave is then

\[
\begin{align*}
\vec{E}(\vec{r},t) &\equiv \text{Re} \left\{ (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k} \cdot \vec{r} - wt)} \right\} \\
\vec{H}(\vec{r},t) &\equiv \frac{\sigma}{\omega \mu} \text{Re} \left\{ k \hat{M} \times (E_1 \hat{e}_1 + E_2 \hat{e}_2) e^{i(\vec{k} \cdot \vec{r} - wt)} \right\} \\
&\equiv \frac{\sigma}{\omega \mu} \text{Re} \left\{ k (E_1 \hat{e}_2 - E_2 \hat{e}_1) e^{i(\vec{k} \cdot \vec{r} - wt)} \right\}
\end{align*}
\]

In general, \( k \) is complex,

\[
k = k_1 + ik_2 = |k| e^{i\delta}, \quad |k| = \sqrt{k_1^2 + k_2^2}, \quad \delta = \text{arctan}(k_2/k_1)
\]

So far we implicitly assumed that \( E_1 \) and \( E_2 \) are real constants. In this case,

\[
\begin{align*}
\vec{E}(\vec{r},t) &\equiv \frac{E_0}{\omega} e^{-k_2 M \cdot \vec{r}} \cos (k_1 \hat{M} \cdot \vec{r} - wt) \\
\vec{H}(\vec{r},t) &\equiv \frac{H_0}{\mu} e^{-k_2 M \cdot \vec{r}} \cos (k_1 \hat{M} \cdot \vec{r} - wt + \delta)
\end{align*}
\]

where

\[
\begin{align*}
E_0 &\equiv E_1 \hat{e}_1 + E_2 \hat{e}_2 \\
H_0 &\equiv \frac{c|k|}{\mu} (E_1 \hat{e}_2 - E_2 \hat{e}_1)
\end{align*}
\]

are fixed vectors for all time and space.
In this case the directions of \( \overrightarrow{E} \) and \( \overrightarrow{H} \) remain fixed while the amplitudes oscillate in time and space. Such a plane wave is called a linearly polarized wave.

However, there is nothing to prevent one from choosing a solution with \( E_1 \) and \( E_2 \) complex numbers,

\[
E_1 = |E_1|e^{ix_1}, \quad E_2 = |E_2|e^{ix_2}
\]

In this case one has

\[
\overrightarrow{E}(\mathbf{r}, t) = k\Re \left\{ |E_1| \hat{e}_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t + x_1)} + |E_2| \hat{e}_2 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t + x_2)} \right\}
\]

\[
= e^{i\mathbf{k} \cdot \mathbf{r} - \omega t} \left[ |E_1| \hat{e}_1 \cos(k_1 \hat{n} \cdot \mathbf{r} - \omega t + x_1) + |E_2| \hat{e}_2 \cos(k_2 \hat{n} \cdot \mathbf{r} - \omega t + x_2) \right]
\]

and

\[
\overrightarrow{H}(\mathbf{r}, t) = \frac{c|k|}{\omega} k \Re \left\{ |E_1| \hat{e}_2 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta + x_1)} - |E_2| \hat{e}_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t + \delta + x_2)} \right\}
\]

\[
= \frac{c|k|}{\omega} e^{-i\mathbf{k} \cdot \mathbf{r} - \omega t} \left[ |E_1| \hat{e}_2 \cos(k_1 \hat{n} \cdot \mathbf{r} - \omega t + \delta + x_1) - |E_2| \hat{e}_1 \cos(k_2 \hat{n} \cdot \mathbf{r} - \omega t + \delta + x_2) \right]
\]

Unless \( x_1 = x_2 \) we see that the components of \( \overrightarrow{E} \) and \( \overrightarrow{H} \) in directions \( \hat{e}_1 \) and \( \hat{e}_2 \) will oscillate out of phase with each other. Thus the directions of \( \overrightarrow{E} \) and \( \overrightarrow{H} \) will oscillate in time and space, as well as the amplitudes of \( \overrightarrow{E} \) and \( \overrightarrow{H} \). The direction of \( \overrightarrow{E} \) and \( \overrightarrow{H} \) is no longer fixed.
We will see that the situation in general corresponds to an elliptically polarized wave.

**General case** $E_1$ and $E_2$ are complex constants

write $E_1 \hat{e}_1 + E_2 \hat{e}_2 = \vec{U} e^{i\gamma}$

where $\gamma$ is chosen so that $\vec{U} \cdot \vec{U}$ is real

- one can always do this since $\vec{U} \cdot \vec{U} = (E_1^2 + E_2^2)e^{-2\gamma}$

so $2\gamma$ is just the phase of the complex $E_1^2 + E_2^2$

$\vec{U}$ is a complex vector $\Rightarrow \vec{U} = \vec{U}_a + z \vec{U}_b$

with $\vec{U}_a$ and $\vec{U}_b$ real vectors

since $\vec{U} \cdot \vec{U}$ is real $\Rightarrow \vec{U}_a \cdot \vec{U}_b = 0$

so $\vec{U}_a \perp \vec{U}_b$ orthogonal

let $\hat{e}_a$ be the unit vector in direction of $\vec{U}_a$

so $\vec{U}_a = U_a \hat{e}_a$ with $U_a = |\vec{U}_a|$

let $\hat{e}_b = \hat{n} \times \hat{e}_a$ so that $\{\hat{n}, \hat{e}_a, \hat{e}_b\}$ are a right-handed coordinate system

Then $\vec{U}_b = \pm U_b \hat{e}_b$ where $U_b = |\vec{U}_b|$

since $\vec{U}_b \perp \vec{U}_a$ and both are $\perp$ to $\hat{n}$.

It is $(\pm)$ if $\vec{U}_b$ is parallel to $\hat{e}_b$ and
it is $(\mp)$ if $\vec{U}_b$ is antiparallel to $\hat{e}_b$. 
In this representation we have

\[ \tilde{E}(\tilde{r}(t)) = \text{Re} \left\{ U_a \hat{e}_a e^{i(k_1 \hat{\vec{r}} \cdot \tilde{r} - \omega t + \Phi)} \pm U_b \hat{e}_b (i) e^{i(k_1 \hat{\vec{r}} \cdot \tilde{r} - \omega t + \Phi)} \right\} \]

\[ = e^{-k_2 \hat{\vec{r}} \cdot \tilde{r}} \left\{ U_a \hat{e}_a \cos(\Phi + \Psi) \pm U_b \hat{e}_b \sin(\Phi + \Psi) \right\} \]

where we write \( \Psi \equiv k_1 \hat{\vec{r}} \cdot \tilde{r} - \omega t \).

Let's define

\[ e^{-k_2 \hat{\vec{r}} \cdot \tilde{r}} U_a \rightarrow U_a \]

\[ e^{-k_2 \hat{\vec{r}} \cdot \tilde{r}} U_b \rightarrow U_b \]

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of \( \tilde{E} \).

Then define \( E_a \) and \( E_b \) as the components of \( \tilde{E} \) in the directions \( \hat{e}_a \) and \( \hat{e}_b \) respectively.

\[ E_a = U_a \cos(\Phi + \Psi) \]

\[ E_b = \mp U_b \sin(\Phi + \Psi) \]

This then gives

\[ \left( \frac{E_a}{U_a} \right)^2 + \left( \frac{E_b}{U_b} \right)^2 = \cos^2(\Phi + \Psi) + \sin^2(\Phi + \Psi) = 1 \]

This is just the equation for an ellipse.
with semi-axes of lengths $U_a$ and $U_b$, oriented in the directions of $\hat{e}_a$ and $\hat{e}_b$.

![Diagram of an ellipse with axes $U_a$ and $U_b$]

$\Rightarrow$ At a fixed position $\mathbf{r}$, the tip of the vector $\mathbf{E}$ will trace out the above ellipse as the time increases by one period of oscillation $2\pi/w$.

For $\mathbf{U}_b = U_b \hat{e}_b$, $\mathbf{E}$ goes around the ellipse counter-clockwise as $t$ increases.

For $\mathbf{U}_b = -U_b \hat{e}_b$, $\mathbf{E}$ goes around the ellipse clockwise as $t$ increases.

Such a wave is said to be elliptically polarized.

**Special cases**

1. $U_a = 0$ or $U_b = 0$
   - The wave is linearly polarized.
2) \( U_a = U_b \)

The tip of \( \vec{E} \) traces out a circle as \( t \) increases. The wave is circularly polarized.

The (+) case is said to have right handed circular polarization.
The (-) case is said to have left handed circular polarization.

One can define circular polarization basis vectors:

\[ \hat{e}_+ = \frac{\hat{e}_a + i \hat{e}_b}{\sqrt{2}} \]
\[ \hat{e}_- = \frac{\hat{e}_a - i \hat{e}_b}{\sqrt{2}} \]

with \( \hat{e}_a \) and \( \hat{e}_b \) orthonormal.

A wave with a complex amplitude \( \vec{E}_w = E \hat{e}_+ + i \hat{e}_- \) is right handed circularly polarized.
A wave with complex amplitude \( \vec{E}_w = E \hat{e}_- + i \hat{e}_+ \) is left handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, i.e.

\[ \vec{E}_w = E_1 \hat{e}_1 + E_2 \hat{e}_2 \]
one can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

\[ \mathbf{U} = \mathbf{U}_a + i \mathbf{U}_b = U_a \hat{e}_a + i U_b \hat{e}_b \]

\[ = \left( \frac{U_a + U_b}{\sqrt{2}} \right) \hat{e}_+ + \left( \frac{U_a - U_b}{\sqrt{2}} \right) \hat{e}_\mp \]

(provide substituting in for \( \hat{e}_\pm \) and expand, to see that this is so)

\[ \Rightarrow \text{ An elliptically polarized wave can be written as a superposition of circularly polarized waves} \]

As a special case of the above (if \( U_a = 0 \) or \( U_b = 0 \)) a linearly polarized wave can always be written as a superposition of circularly polarized waves,
magnetic field

In the above general formulation, we can write $\vec{H}$ as

$$\vec{H} = \frac{c|k|}{\omega \mu} \text{Re} \left\{ k \hat{m} \times U e^{i\Psi} e^{i(k \cdot \hat{r} - \omega t)} \right\}$$

$$= \frac{c|k|}{\omega \mu} \text{Re} \left\{ \hat{m} \times (U_a \hat{e}_a \pm iU_b \hat{e}_b) e^{i(k \cdot \hat{r} - \omega t + \delta + \varepsilon)} \right\}$$

$$= \frac{c|k|}{\omega \mu} \text{Re} \left\{ (U_a \hat{e}_b - iU_b \hat{e}_a) e^{i(k \cdot \hat{r} - \omega t + \delta + \varepsilon)} \right\}$$

$$\vec{H} = \frac{c|k|}{\omega \mu} e^{-k \cdot \hat{m} \cdot \hat{r}} \left[ -U_a \hat{e}_b \cos (\Phi + \Psi + \varepsilon) \pm U_b \hat{e}_a \sin (\Phi + \Psi + \varepsilon) \right]$$

we had for the electric field

$$\vec{E} = e^{-k \cdot \hat{m} \cdot \hat{r}} \left[ U_a \hat{e}_a \cos (\Phi + \Psi) \pm U_b \hat{e}_b \sin (\Phi + \Psi) \right]$$

Consider $\vec{E} \cdot \vec{H}$. From the above, with $\hat{e}_a \cdot \hat{e}_b = 0$, we get

$$\vec{E} \cdot \vec{H} = e^{-2k \cdot \hat{m} \cdot \hat{r}} \frac{c|k|}{\omega \mu} U_a U_b (\pm 1) \left[ \sin (\Phi + \Psi + \varepsilon) \cos (\Phi + \Psi) - \cos (\Phi + \Psi + \varepsilon) \sin (\Phi + \Psi) \right]$$

$$= e^{-2k \cdot \hat{m} \cdot \hat{r}} \frac{c|k|}{\omega \mu} U_a U_b (\pm 1) \sin \delta$$

where in the last step we used $\sin A \cos B - \cos A \sin B = \sin (A - B)$

We see that $\vec{E} \cdot \vec{H} = 0$ only when

1) $\delta = 0$, i.e., the medium has no dispersion

or

2) $U_a = 0$ or $U_b = 0$, i.e., the wave is linearly polarized