The main difference between dielectrics & conductors has to do with the contribution that the $4\pi i\sigma/\omega$ makes to the real and imaginary parts of $\varepsilon(\omega)$.

For single Drude model $\sigma(\omega) = \dfrac{\sigma_0}{1-i\omega\tau}$ $\quad \sigma_0 = \dfrac{ne^2\tau}{m}$

① Low frequencies $\omega \ll 1/\tau$

$\varepsilon_b(\omega) \approx \varepsilon_b(0)$ real

$\sigma(\omega) \approx \sigma_0$ real

$\Rightarrow \boxed{\varepsilon(\omega) \approx \varepsilon_b(0) + \dfrac{4\pi i\sigma_0}{\omega}}$ ← gives large ε_2 as $\omega \to 0$

⇒ strong dissipation

$\mathrm{Re}\,\varepsilon = \varepsilon_1$

$\mathrm{Im}\,\varepsilon = \varepsilon_2$

when $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{4\pi\sigma_0}{\omega\varepsilon_b(0)} \gg 1$ we call this regime a "good" conductor.
conduction electrons dominate the response
– waves strongly attenuated

when $\dfrac{\varepsilon_2}{\varepsilon_1} = \dfrac{4\pi\sigma_0}{\omega\,\varepsilon_b(0)} \ll 1$ we call this regime a "poor" conductor.
little absorption of energy by conduction electrons.
waves propagate

One always enters the "good" conductor region when ω gets sufficiently small.

wave vector:

$$k = \frac{\omega}{c}\sqrt{\mu\varepsilon}$$

for a good conductor where $\varepsilon_2 \gg \varepsilon_1$,

$$\varepsilon \sim i\varepsilon_2 = \frac{4\pi i \sigma_0}{\omega}$$

$$k = k_1 + i k_2 = \frac{\omega}{c}\sqrt{\mu \frac{4\pi i \sigma_0}{\omega}} \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

for $\vec{k} = k\hat{z}$,

$$\vec{E} = \vec{E}_\omega e^{i(kz-\omega t)} = \vec{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$$

$$\delta \equiv 1/k_2 = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}}$$

"skin depth"
distance wave propagates into conductor

$\delta \sim 1/\sqrt{\omega}$ increases as ω decreases

ϕ phase shift between oscillations of $\vec{E}$ and $\vec{H}$

$$\phi = \arctan(k_2/k_1) \simeq \arctan(1) = 45°$$

Amplitude ratio

$$\frac{|\vec{H}_\omega|}{|\vec{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}\,c}{\omega\mu} k_1$$

$$= \frac{\sqrt{2}\,c}{\omega\mu}\frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega}$$

$$= \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim 1/\sqrt{\omega}$$

as $\omega \to 0$, most of the energy of the wave is carried by the magnetic field part

② <u>high frequencies</u> $\omega \gg 1/\tau$, $\omega \gg \omega_0$

$\varepsilon_b(\omega) \approx 1$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{i n e^2 \tau}{m\omega\tau} = \frac{i n e^2}{m\omega}$$ pure imaginary indep of τ

$$\varepsilon(\omega) \simeq 1 + \frac{4\pi i \sigma}{\omega} \simeq 1 - \frac{4\pi n e^2}{m\omega^2}$$

$$\boxed{\varepsilon(\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2}}$$

$\omega_p \equiv \sqrt{\frac{4\pi n e^2}{m}}$ plasma freq of the conduction electrons

$\varepsilon(\omega)$ is <u>real</u>

1) If $\omega > \omega_p$ then $\varepsilon > 0$

$\Rightarrow$ transparent propagation

$k = k_1 = \frac{\omega}{c}\sqrt{\mu\varepsilon}$ is pure real

$k_2 \approx 0$

2) If $\omega < \omega_p$ then $\varepsilon < 0$

$\Rightarrow$ total reflection

$k_1 \simeq 0$

$k = k_2 = \frac{\omega}{c}\sqrt{\mu|\varepsilon|}$ k is pure imaginary

ω_p gives cross over between total reflection and transparent propagation for typical metals

$$\tau \sim 10^{-14} \text{ sec}$$

$$\omega_p \sim 10^{16} \text{ sec}^{-1}$$

$$\lambda_p = \frac{2\pi c}{\omega_p} \sim 3\times10^3 \text{ Å} \quad (\text{visible is } \lambda \sim 5\times10^3 \text{ Å})$$

<u>Example</u>: The ionosphere is a layer of charged gas surrounding the earth. In many respects the charged particles of the ionosphere behave like conduction electrons in a metal. The plasma freq of the ionosphere is such that

for AM radio $\omega_{AM} < \omega_p \Rightarrow$ AM radio signals reflected back to earth

for FM radio $\omega_{FM} > \omega_p \Rightarrow$ FM radio signals propagate through ionosphere into space

Explains why you can pick up AM stations from far away - they get reflected back. But you can only pick up local FM stations.

Longitudinal modes in conductors

ie $\vec{H}_\omega$ or $\vec{E}_\omega$ not $\perp \vec{k}$

magnetic field

$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow i\mu \vec{k}\cdot\vec{H}_\omega = 0 \Rightarrow \vec{H}_\omega \perp \vec{k}$ transverse

or $\vec{k}=0$ spatially uniform $\vec{H}$

if $\vec{k}=0$ then Faraday

$$\underbrace{i\vec{k}\times\vec{E}_\omega}_{0 \text{ as } \vec{k}=0} = \frac{i\omega\mu}{c}\vec{H}_\omega = 0 \Rightarrow \omega = 0$$

So only possible longitudinal $\vec{H}$ is spatially uniform, constant in time.

electric field

$\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f \Rightarrow i\varepsilon(\omega)\vec{k}\cdot\vec{E}_\omega = 0 \Rightarrow \vec{E}_\omega \perp \vec{k}$ transverse

or $\varepsilon(\omega)=0$

If $\vec{E}_\omega \parallel \vec{k}$ but $\varepsilon(\omega)=0$, then can satisfy all other maxwell equations.

$$i\vec{k}\times\vec{E}_\omega = \frac{i\omega\mu}{c}\vec{H}_\omega \Rightarrow \vec{H}_\omega = 0$$

$$\Rightarrow \underbrace{i\mu_0\vec{k}\cdot\vec{H}_\omega}_{0 \text{ as } \vec{H}_\omega=0} = 0 \quad \text{and} \quad i\vec{k}\times\vec{H}_\omega = \underbrace{-\frac{i\omega}{c}\varepsilon(\omega)\vec{E}_\omega}_{0 \text{ as } \varepsilon(\omega)=0}$$

So we can have longitudinal electric field oscillation when $\varepsilon(\omega)=0$

low freq $\quad \omega \ll \omega_0 \quad \omega\tau \ll 1$

$$\varepsilon \simeq \varepsilon_b(0) + \frac{4\pi i \sigma_0}{\omega}$$

$$\varepsilon(\omega) = 0 \quad \text{when} \quad \omega = -\frac{4\pi i \sigma_0}{\varepsilon_b(0)}$$

$$\vec{E}(\vec{r},t) = \vec{E}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_\omega e^{-\frac{4\pi\sigma_0}{\varepsilon_b(0)}t} e^{i\vec{k}\cdot\vec{r}}$$

If set up a longitudinal $\vec{E}$ field, it decays to zero exponentially with ~~time const~~ decay time $\varepsilon_b(0)/4\pi\sigma_0$

This is consistant with assumption the $\vec{E}=0$ inside a conductor for electrostatics

in statics $\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{E} \sim -i\vec{k}\phi_k e^{i\vec{k}\cdot\vec{r}}$ is longitude

high freq $\quad \omega \gg 1/\tau, \ \omega \gg \omega_0$

$$\varepsilon(\omega) \approx 1 + \frac{4\pi i\sigma}{\omega} = 1 - \frac{\omega_p^2}{\omega^2} \qquad \omega_p^2 = \frac{4\pi n e^2}{m}$$

$\varepsilon = 0$ when $\omega = \omega_p$

So we have oscillatory longitudinal $\vec{E}$ only when $\omega = \omega_p$, independent of $\vec{k}$.

$$\vec{E} = \vec{E}_\omega e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t}$$

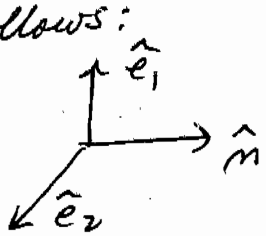
This is called a plasma oscillation. When one quantizes this oscillatory mode, it is called a plasmon

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \rho = \frac{i\vec{k}\cdot\vec{E}_\omega}{4\pi} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_p t} \Leftarrow$$

plasma osc. is a charge density oscillation

Polarization

Consider a transverse plane wave traveling in direction $\hat{n}$, ie $\vec{k} = k\hat{n}$. Define a right handed coordinate system as follows:



$$\hat{e}_1 \times \hat{e}_2 = \hat{n}$$
$$\hat{n} \times \hat{e}_1 = \hat{e}_2$$
$$\hat{e}_2 \times \hat{n} = \hat{e}_1$$

A general solution to Maxwell equations for a transverse plane wave is then

$$\vec{E}(\vec{r},t) = Re\left\{ (E_1\hat{e}_1 + E_2\hat{e}_2)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$
$$\vec{H}(\vec{r},t) = \frac{c}{\omega\mu} Re\left\{ k\,\hat{n}\times(E_1\hat{e}_1 + E_2\hat{e}_2)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$
$$= \frac{c}{\omega\mu} Re\left\{ k\,(E_1\hat{e}_2 - E_2\hat{e}_1)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

In general, k is complex

$$k = k_1 + ik_2 = |k|\,e^{i\delta}, \quad \begin{cases} |k| = \sqrt{k_1^2 + k_2^2} \\ \delta = \arctan(k_2/k \end{cases}$$

So far we implicitly assumed that E_1 and E_2 are real constants. In this case

$$\vec{E}(\vec{r},t) = \vec{E}_\omega\, e^{-k_2\hat{n}\cdot\vec{r}} \cos(k_1\hat{n}\cdot\vec{r} - \omega t)$$
$$\vec{H}(\vec{r},t) = \vec{H}_\omega\, e^{-k_2\hat{n}\cdot\vec{r}} \cos(k_1\hat{n}\cdot\vec{r} - \omega t + \delta)$$

where

$$\vec{E}_\omega = E_1\hat{e}_1 + E_2\hat{e}_2 \quad \text{and} \quad \vec{H}_\omega \equiv \frac{c|k|}{\omega\mu}(E_1\hat{e}_2 - E_2\hat{e}_1)$$

are *fixed* vectors for all time and space.

In this case the directions of $\vec{E}$ and $\vec{H}$ remain fixed while the amplitudes oscillate in time and space. Such a plane wave is called a linearly polarized wave.

However there is nothing to prevent one from choosing a solution with E_1 and E_2 complex numbers,

$$E_1 = |E_1| e^{i\chi_1}, \quad E_2 = |E_2| e^{i\chi_2}$$

In this case one has

$$\vec{E}(\vec{r},t) = Re\left\{ |E_1| \hat{e}_1 e^{i(\vec{k}\cdot\vec{r} - \omega t + \chi_1)} + |E_2| \hat{e}_2 e^{i(\vec{k}\cdot\vec{r} - \omega t + \chi_2)} \right\}$$

$$= e^{-k_2 \hat{m}\cdot\vec{r}} \Big[|E_1| \hat{e}_1 \cos(k_1 \hat{m}\cdot\vec{r} - \omega t + \chi_1) + |E_2| \hat{e}_2 \cos(k_1 \hat{m}\cdot\vec{r} - \omega t + \chi_2) \Big]$$

and

$$\vec{H}(\vec{r},t) = \frac{c|k|}{\omega\mu} Re\left\{ |E_1| \hat{e}_2 e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta + \chi_1)} - |E_2| \hat{e}_1 e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta + \chi_2)} \right\}$$

$$= \frac{c|k|}{\omega\mu} e^{-k_2 \hat{m}\cdot\vec{r}} \Big[|E_1| \hat{e}_2 \cos(k_1 \hat{m}\cdot\vec{r} - \omega t + \delta + \chi_1) - |E_2| \hat{e}_1 \cos(k_1 \hat{m}\cdot\vec{r} - \omega t + \delta + \chi_2) \Big]$$

Unless $\chi_1 = \chi_2$ we see that the components of $\vec{E}$ and $\vec{H}$ in directions $\hat{e}_1$ and $\hat{e}_2$ will oscillate out of phase with each other. Thus the directions of $\vec{E}$ and $\vec{H}$ will oscillate in time and space, as well as the amplitudes of $\vec{E}$ and $\vec{H}$. The direction of $\vec{E}$ and $\vec{H}$ is no longer fixed.

We will see that this situation in general corresponds to an elliptically polarized wave!

General case $\quad E_1$ and E_2 are complex constants

write $E_1\hat{e}_1 + E_2\hat{e}_2 \equiv \vec{u}\, e^{i\psi}$

where ψ is chosen so that $\vec{u}\cdot\vec{u}$ is real

– one can always do this since $\vec{u}\cdot\vec{u} = (E_1^2+E_2^2)e^{-2i\psi}$

so 2ψ is just the phase of the complex $E_1^2+E_2^2$

$\vec{u}$ is a complex vector $\Rightarrow \vec{u} = \vec{u}_a + i\vec{u}_b$

with $\vec{u}_a$ and $\vec{u}_b$ real vectors

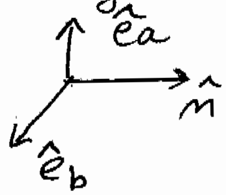
Since $\vec{u}\cdot\vec{u}$ is real $\Rightarrow \vec{u}_a\cdot\vec{u}_b = 0$

so $\vec{u}_a \perp \vec{u}_b$ orthogonal

Let $\hat{e}_a$ be the unit vector in direction of $\vec{u}_a$

so $\vec{u}_a = u_a\hat{e}_a \quad$ with $u_a = |\vec{u}_a|$

Let $\hat{e}_b = \hat{n}\times\hat{e}_a$ so that $\{\hat{n}, \hat{e}_a, \hat{e}_b\}$ are a right handed coordinate system



Then $\vec{u}_b = \pm u_b\hat{e}_b$ where $u_b = |\vec{u}_b|$

since $\vec{u}_b \perp \vec{u}_a$ and both are $\perp$ to $\hat{n}$.

It is $(+)$ if $\vec{u}_b$ is parallel to $\hat{e}_b$ and it is $(-)$ if $\vec{u}_b$ is antiparallel to $\hat{e}_b$.

In this representation we have

$$\vec{E}(\vec{r},t) = Re\left\{ \vec{u}\, e^{i\psi} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}$$

$$= e^{-k_2\hat{m}\cdot\vec{r}}\, Re\left\{ u_a \hat{e}_a e^{i(k_1\hat{m}\cdot\vec{r}-\omega t+\psi)} \pm u_b \hat{e}_b (+i) e^{i(k_1\hat{m}\cdot\vec{r}-\omega t+\psi)} \right\}$$

$$= e^{-k_2\hat{m}\cdot\vec{r}} \left\{ u_a \hat{e}_a \cos(\Phi+\psi) \mp u_b \hat{e}_b \sin(\Phi+\psi) \right\}$$

where we write $\Phi \equiv k_1 \hat{m}\cdot\vec{r} - \omega t$

Let's define $e^{-k_2\hat{m}\cdot\vec{r}} u_a \rightarrow u_a$

$e^{-k_2\hat{m}\cdot\vec{r}} u_b \rightarrow u_b$

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of $\vec{E}$.

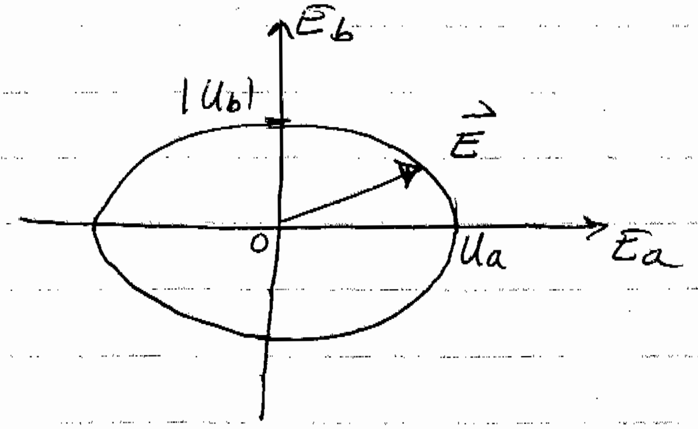
Then define E_a and E_b as the components of $\vec{E}$ in the directions $\hat{e}_a$ and $\hat{e}_b$ respectively.

$$E_a = u_a \cos(\Phi+\psi)$$
$$E_b = \mp u_b \sin(\Phi+\psi)$$

This then gives

$$\left(\frac{E_a}{u_a}\right)^2 + \left(\frac{E_b}{u_b}\right)^2 = \cos^2(\Phi+\psi) + \sin^2(\Phi+\psi) = \underline{1}$$

This is just the equation for an ellipse with semiaxes of lengths U_a and U_b, oriented in the directions of $\hat{e}_a$ and $\hat{e}_b$.



$\Rightarrow$ At a fixed position $\vec{r}$, the tip of the vector $\vec{E}$ will trace out the above ellipse as the time increases by one period of oscillation $2\pi/\omega$.

For (+), ie $\vec{U}_b = U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>counterclockwise</u> as t increases

For (−), ie $\vec{U}_b = -U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>clockwise</u> as t increases

Such a wave is said to be <u>elliptically polarized</u>

<u>Special cases</u>

① $U_a = 0$ or $U_b = 0$
the wave is <u>linearly polarized</u>

(2) $u_a = u_b$

The tip of $\vec{E}$ traces out a ~~circle~~ circle as t increases. The wave is circularly polarized.

The (+) case is said to have right handed circular polarization

The (−) case is said to have left handed circular polarization

One can define circular polarization basis vectors

$$\hat{e}_+ \equiv \frac{\hat{e}_a + i\hat{e}_b}{\sqrt{2}} \qquad \hat{e}_- \equiv \frac{\hat{e}_a - i\hat{e}_b}{\sqrt{2}}$$

with $\hat{e}_a$ and $\hat{e}_b$ orthogonal.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_+$ is right handed circularly polarized.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_-$ is left handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, ie

$$\vec{E}_\omega = E_1\hat{e}_1 + E_2\hat{e}_2$$

One can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

$$\vec{U} = \vec{U}_a + i\vec{U}_b = U_a \hat{e}_a \pm i U_b \hat{e}_b$$

$$= \left(\frac{U_a + U_b}{\sqrt{2}}\right)\hat{e}_\pm + \left(\frac{U_a - U_b}{\sqrt{2}}\right)\hat{e}_\mp$$

(substitute in for $\hat{e}_\pm$ and expand, to see that this is so)

$\Rightarrow$ An elliptically polarized wave can be written as a superposition of circularly polarized waves

As a special case of the above (if $U_a = 0$ or $U_b = 0$) a <u>linearly</u> polarized wave can always be written as a superposition of circularly polarized waves.

magnetic field

In the above general formulation we can write $\vec{H}$ as

$$\vec{H} = \frac{c}{\omega\mu} \mathrm{Re}\left\{ k\hat{m} \times \vec{U} e^{i\psi} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

$$= \frac{c|k|}{\omega\mu} \mathrm{Re}\left\{ \hat{m} \times (U_a \hat{e}_a \pm i U_b \hat{e}_b) e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta + \psi)} \right\}$$

$$= \frac{c|k|}{\omega\mu} \mathrm{Re}\left\{ (U_a \hat{e}_b \mp i U_b \hat{e}_a) e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta + \psi)} \right\}$$

$$\vec{H} = \frac{c|k|}{\omega\mu} e^{-k_2 \hat{m}\cdot\vec{r}} \Big[U_a \hat{e}_b \cos(\Phi + \psi + \delta) \pm U_b \hat{e}_a \sin(\Phi + \psi + \delta) \Big]$$

we had for the electric field

$$\vec{E} = e^{-k_2 \hat{m}\cdot\vec{r}} \Big[U_a \hat{e}_a \cos(\Phi + \psi) \mp U_b \hat{e}_b \sin(\Phi + \psi) \Big]$$

Consider $\vec{E}\cdot\vec{H}$. From the above, with $\hat{e}_a \cdot \hat{e}_b = 0$, we get

$$\vec{E}\cdot\vec{H} = e^{-2k_2 \hat{m}\cdot\vec{r}} \frac{c|k|}{\omega\mu} U_a U_b (\pm 1) \Big[\sin(\Phi + \psi + \delta)\cos(\Phi + \psi) - \cos(\Phi + \psi + \delta)\sin(\Phi + \psi) \Big]$$

$$= e^{-2k_2 \hat{m}\cdot\vec{r}} \frac{c|k|}{\omega\mu} U_a U_b (\pm 1) \sin\delta$$

where in the last step we used $\sin A \cos B - \cos A \sin B = \sin(A - B)$

We see that $\vec{E}\cdot\vec{H} = 0$ only when

1) $\delta = 0$, ie the medium has no dissipation

or

2) $U_a = 0$ or $U_b = 0$, ie the wave is <u>linearly polarized</u>