

Kramers - Kronig Relation

We saw that $\vec{F}_\omega = \alpha(\omega) \vec{E}_\omega$

Causal response is $\tilde{\alpha}(t) = 0$ for $t < 0$

$\Rightarrow \alpha(\omega)$ has no poles in upper half of complex ω plane (UHP)

For any complex $\bar{\omega}$ in upper half of complex ω plane,

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i} \oint \frac{\alpha(\omega')}{\omega' - \bar{\omega}} d\omega' \quad \text{since no poles of } \alpha \text{ in UHP}$$

\Rightarrow contour along real axis, closed at infinity in UHP. The closing ~~top~~ semicircle at infinity gives no contribution assuming $\alpha(\omega)$ decays quickly enough as $|\omega| \rightarrow \infty$

$$\alpha(\bar{\omega}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \bar{\omega}}$$

Now consider $\bar{\omega} = \omega + i\delta$ where ω and δ are real and $\delta \rightarrow 0$

$$\alpha(\omega) = \lim_{\delta \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \frac{\alpha(\omega')}{\omega' - \omega - i\delta}$$

$$\text{Now } \frac{1}{\omega' - \omega - i\delta} = P\left(\frac{1}{\omega' - \omega}\right) + i\pi\delta(\omega' - \omega)$$

\uparrow principle part

$$\Rightarrow \alpha(\omega) = \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} \frac{\alpha(\omega') d\omega'}{\omega' - \omega}$$

$$\Rightarrow \left. \begin{aligned} \operatorname{Re} \alpha(\omega) &= \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \alpha(\omega') d\omega'}{\omega' - \omega} \\ \operatorname{Im} \alpha(\omega) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\operatorname{Re} \alpha(\omega') d\omega'}{\omega' - \omega} \end{aligned} \right\} \begin{array}{l} \text{Kramer} \\ \text{Kronig} \\ \text{relations} \end{array}$$

If know $\operatorname{Re} \alpha$ or $\operatorname{Im} \alpha$ can reconstruct full complex α

True for any causal response function

Radiation From moving Charges

In Lorentz gauge $\frac{1}{c} \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$

$$\left. \begin{aligned} \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} &= -4\pi \rho \\ \nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} &= -\frac{4\pi}{c} \vec{j} \end{aligned} \right\} \text{wave equation with source}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

If we can solve wave equation with source (inhomogeneous wave equation) then we are in principle done! To do this we want to find the Green's function for the wave equation

Recall from statics: $\nabla^2 \phi = -4\pi \rho$

Green's function satisfies $\nabla^2 G(\vec{r}) = -4\pi \delta(\vec{r})$

$$\text{Then } \phi(\vec{r}) = \int d^3 r' G(\vec{r} - \vec{r}') \rho(\vec{r}') + \phi_0$$

solution for infinite volume that vanishes as $r \rightarrow \infty$ is

$$G(\vec{r} - \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} \quad \nabla^2 \phi_0 = 0$$

For wave equation we want solution to

$$\nabla^2 G(\vec{r}, t; \vec{r}', t') - \frac{1}{c^2} \frac{\partial^2 G(\vec{r}, t; \vec{r}', t')}{\partial t^2} = -4\pi \delta(\vec{r} - \vec{r}') \delta(t - t')$$

$$\text{then we will have } \left\{ \begin{aligned} \phi(\vec{r}, t) &= \int d^3 r' G(\vec{r}, t; \vec{r}', t') \rho(\vec{r}', t') + \phi_0 \\ \vec{A}(\vec{r}, t) &= \frac{1}{c} \int d^3 r' G(\vec{r}, t; \vec{r}', t') \vec{j}(\vec{r}', t') + \vec{A}_0 \end{aligned} \right.$$

where $\nabla^2 \phi_0 - \frac{1}{c^2} \frac{\partial^2 \phi_0}{\partial t^2} = 0$ similarly for \vec{A}_0

ϕ_0 or \vec{A}_0 could describe an incoming wave for example
 To construct the Green's function.

For infinite space (but not, for example, inside a cavity)

$$G(\vec{r}_0; \vec{r}'_0, t-t') = G(\vec{r}-\vec{r}', t-t')$$

express as Fourier transform

$$G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\equiv \int \frac{d^3k d\omega}{(2\pi)^4} \tilde{G}(\vec{k}, \omega) \left[-k^2 + \frac{\omega^2}{c^2}\right] e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$-4\pi \delta(\vec{r}) \delta(t) = -4\pi \int \frac{d^3k d\omega}{(2\pi)^4} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

equate Fourier amplitudes

$$\Rightarrow \left[-k^2 + \frac{\omega^2}{c^2}\right] \tilde{G}(\vec{k}, \omega) = -4\pi$$

$$\tilde{G}(\vec{k}, \omega) = \frac{4\pi c^2}{k^2 c^2 - \omega^2}$$

when $\omega^2 \neq c^2 k^2$

$$G(\vec{r}, t) = \int \frac{d^3k d\omega}{(2\pi)^4} \frac{4\pi c^2}{k^2 c^2 - \omega^2} e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

↑ poles at $\omega = \pm ck$

In evaluating the ω integral we have to know how

to treat the poles on the real axis so that $G(\vec{r}, t)$ will have the desired behavior.

What we want is for $G(\vec{r}, t)$ to be causal, i.e. $G(\vec{r}, t) = 0$ for $t < 0$, so $\phi(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$ depend only on the values of the sources at earlier times $t' < t$.

$$\int d^3k e^{i\vec{k}\cdot\vec{r}} \tilde{G}(\vec{k}, \omega) = 2\pi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk k^2 e^{ikr \cos\theta} \tilde{G}(k, \omega)$$

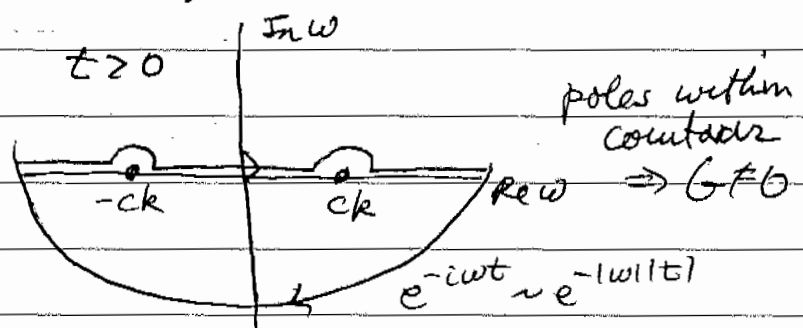
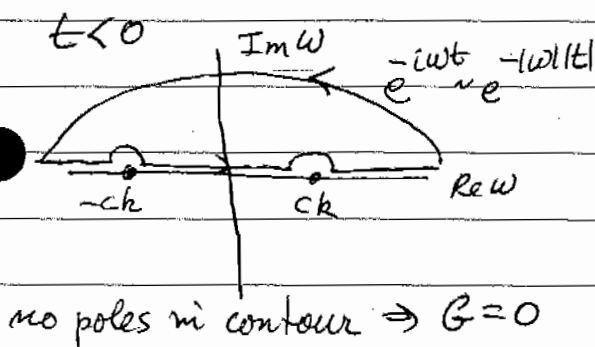
$$= 2\pi \int_{-1}^1 d\mu \int_0^{\infty} dk k^2 e^{ikr\mu} \tilde{G}(k, \omega) \quad \mu \equiv \cos\theta$$

$$= 4\pi \int_0^{\infty} dk k^2 \frac{\sin kr}{kr} \tilde{G}(k, \omega)$$

$$G(\vec{r}, t) = -\frac{c^2}{4\pi} \int_0^{\infty} dk k^2 \frac{\sin kr}{kr} \int_C \frac{e^{-i\omega t}}{(\omega + ck)(\omega - ck)} d\omega$$

contour along real axis, but deformed to go around the poles

for $t < 0$, $e^{-i\omega t}$ will decay exponentially fast for large $|\omega|$ in the upper half complex (UHP) ω plane \Rightarrow can close contour in UHP for $t < 0$. If we want $G = 0$ for $t < 0$, there should therefore be no poles in UHP. The contour C we want is therefore:



with this convention for the contour C we can evaluate the ω -integral using Cauchy's residue theorem

$$\int \frac{e^{-i\omega t}}{(\omega+ck)(\omega-ck)} d\omega = -2\pi i \left[\frac{e^{-ickt}}{2ck} - \frac{e^{ickt}}{2ck} \right]$$

$$= -\frac{2\pi \sin(ckt)}{ck}$$

$$G(\vec{r}, t) = \frac{2c}{\pi r} \int_0^{\infty} dk \sin(kr) \sin(ckt) = \frac{c}{\pi r} \int_{-\infty}^{\infty} dk \frac{(e^{ikr} - e^{-ikr})(e^{ickt} - e^{-ickt})}{(-4)}$$

$$= -\frac{c}{2r} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[e^{i(r+ct)k} + e^{-i(r+ct)k} - e^{i(r-ct)k} - e^{-i(r-ct)k} \right]$$

each integral would give a δ -function, but for 1st two terms $\delta(r+ct) = 0$ since here $t \geq 0$ (by definition) and $r = |\vec{r}| \geq 0$ so the argument will never vanish.

$$G(\vec{r}, t) = \frac{c}{r} \delta(r-ct) = \frac{\delta(t - r/c)}{r} \quad \text{using } \delta(ax) = \frac{1}{|a|} \delta(x)$$

$$G(\vec{r}, t, \vec{r}', t') = \begin{cases} \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \left. \begin{array}{l} \text{Green's function} \\ \text{for wave equation} \\ \text{in free space} \end{array} \right\}$$

$G \neq 0$ only on "light cone" that emanates from (\vec{r}', t') , i.e. when $|\vec{r}-\vec{r}'| = c(t-t')$.
Signal from source at (\vec{r}', t') travels with c