

$$\phi(\vec{r}, t) = \phi_0(\vec{r}, t) + \int_{-\infty}^t d^3r' \int dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \rho(\vec{r}', t')$$

$$A(\vec{r}, t) = \vec{A}_0(\vec{r}, t) + \int_{-\infty}^t d^3r' \int dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \vec{j}(\vec{r}', t')$$

Apply to a single moving point charge

$$\rho(\vec{r}, t) = q \delta(\vec{r} - \vec{r}_0(t))$$

$$\vec{j}(\vec{r}, t) = q \vec{v}(t) \delta(\vec{r} - \vec{r}_0(t)) \quad \text{where } \vec{v}(t) = \frac{d\vec{r}_0}{dt}$$

Then

$$\phi(\vec{r}, t) = q \int dt' \frac{\delta(t-t' - \frac{1}{c} |\vec{r} - \vec{r}_0(t')|)}{|\vec{r} - \vec{r}_0(t')|}$$

because of the $\vec{r}_0(t')$ in the argument of the $\delta()$ function the t' dependence is not of the simple form $t' - t_0$.

We can write

$$g(t') \equiv t' + \frac{1}{c} |\vec{r} - \vec{r}_0(t')|$$

Then

$$\phi(\vec{r}, t) = q \int dt' \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|}$$

$$= q \int \frac{\delta(t - g(t'))}{|\vec{r} - \vec{r}_0(t')|} dq \left(\frac{dt'}{dq} \right)$$

$$= \frac{q}{|\vec{r} - \vec{r}_0(t')|} \frac{1}{(dq/dt')} \Big|_{t' \text{ such that } g(t') = t}$$

$$g(t') = t' + \frac{1}{c} \sqrt{[x-x_0(t')]^2 + [y-y_0(t')]^2 + [z-z_0(t')]^2}$$

$$\frac{dg}{dt'} = 1 + \frac{1}{c |\vec{r}-\vec{r}_0(t')|} \left\{ [x-x_0(t')] \left(-\frac{dx_0}{dt'} \right) + \dots \right\}$$

$$= 1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t')$$

where $\hat{n}(t') = \frac{\vec{r}-\vec{r}_0(t')}{|\vec{r}-\vec{r}_0(t')|}$

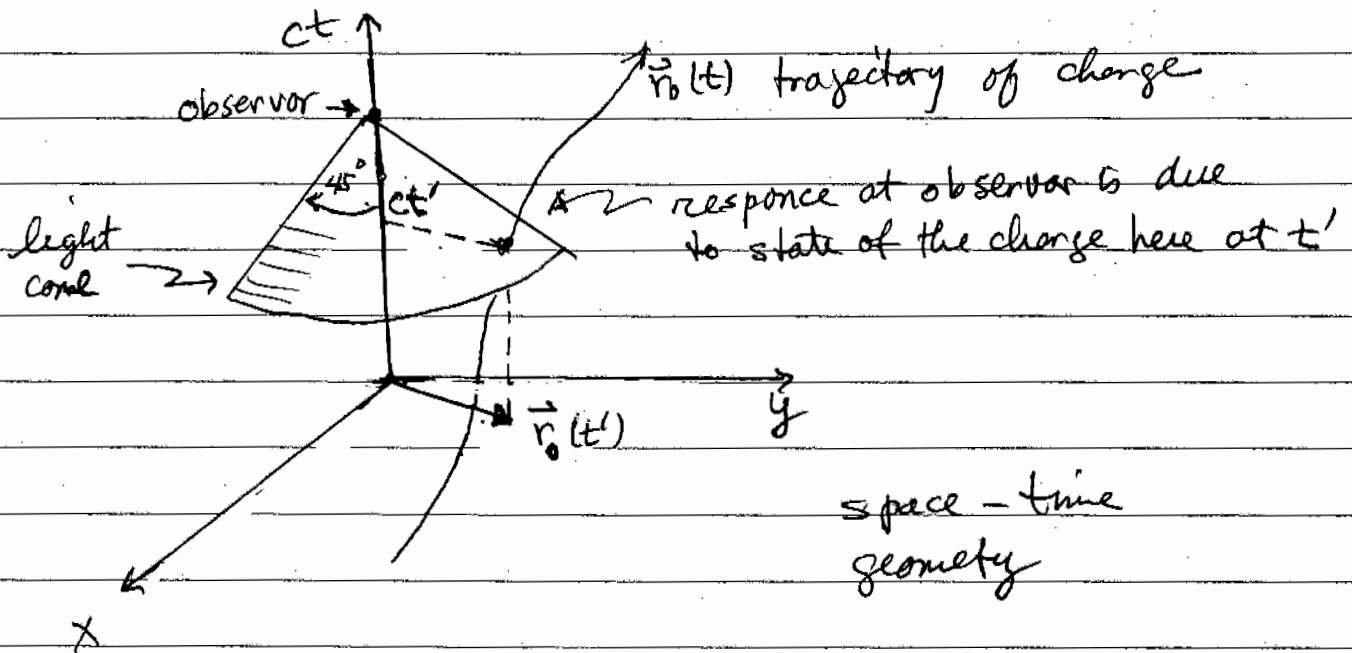
$$\phi(\vec{r}, t) = \frac{q}{|\vec{r}-\vec{r}_0(t')| \left[1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t') \right]}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}(t')/c}{|\vec{r}-\vec{r}_0(t')| \left[1 - \frac{1}{c} \hat{n}(t') \cdot \vec{v}(t') \right]}$$

Liénard
-Wiechert
Potentials

where t' is determined by the condition

$$t - t' = \frac{1}{c} |\vec{r}-\vec{r}_0(t')|$$



For charge moving with constant velocity along \hat{z}

$$\vec{r}_0(t) = vt \hat{z} \quad \vec{v} = \frac{d\vec{r}_0}{dt} = v \hat{z}$$

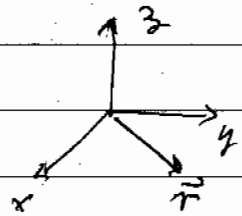
(in xy plane)

For observer at position \vec{r} , time t , the fields will be determined by the charge at time t' such that

$$t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c} = 0$$

$$t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0$$

$$(t - t')^2 = t^2 + t'^2 - 2tt' = \frac{r^2 + v^2 t'^2}{c^2}$$



$$(1 - v^2/c^2) t'^2 - 2tt' + t^2 - \frac{r^2}{c^2} = 0$$

$$\text{let } \gamma = (1 - v^2/c^2)^{-1/2}$$

$$t'^2 - 2\gamma t t' + \gamma^2 (c^2 t^2 - r^2) = 0$$

$$t' = \gamma^2 t \pm \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 \frac{r^2}{c^2}}$$

$$= \gamma^2 t \pm \sqrt{\gamma^2 (\gamma^2 t^2 - t^2 + \frac{r^2}{c^2})}$$

$$\gamma^2 - 1 = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 - v^2/c^2}$$

$$= \gamma^2 t \pm \gamma \sqrt{t^2 \gamma^2 \left(\frac{v^2}{c^2} \right) + \frac{r^2}{c^2}}$$

$$t' = \gamma^2 t \pm \frac{\gamma^2}{c} \sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

consider $t=0$. solution should give $t' < 0$
 $\Rightarrow (-)$ sign is the solution we want

$$t' = \gamma^2 t - \frac{\gamma^2}{c} \sqrt{v^2 t^2 + r^2}$$

$$\phi(\vec{r}, t) = \frac{q}{|\vec{r} - \vec{r}_0(t')| \left[1 - \frac{1}{c} \dot{M}(t') \cdot \vec{v} \right]}$$

~~$$\vec{r} - \vec{r}_0(t') = \vec{r} - \vec{v} t'$$~~

$$|\vec{r} - \vec{r}_0(t')| = \sqrt{r^2 + v^2 t'^2} = c(t - t') \quad \leftarrow \text{from condition that determines } t'$$

$$(\vec{r} - \vec{r}_0(t')) \cdot \vec{v} = -\vec{r}_0(t') \cdot \vec{v} \quad \text{for } \vec{v} = v \hat{z}$$

$$= -v^2 t' \quad \vec{r} \text{ in } xy \text{ plane}$$

$$\phi(\vec{r}, t) = \frac{q}{c(t - t') \left[1 + \frac{v^2 t'}{c^2(t - t')} \right]}$$

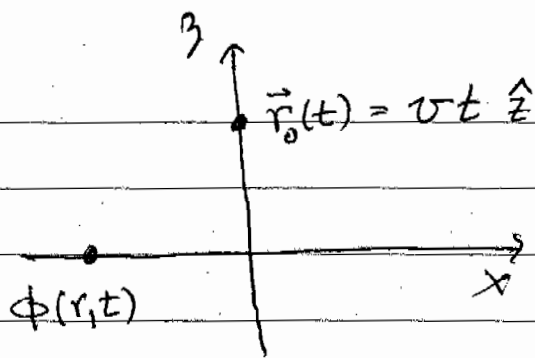
$$= \frac{q}{c(t - t') + \frac{v^2 t'}{c}} = \frac{q}{c \left[t - \left(1 - \frac{v^2}{c^2}\right) t' \right]}$$

$$= \frac{q}{c \left(t - \frac{t'}{\gamma^2} \right)} = \frac{q}{c \frac{1}{c} \sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

$$\phi(\vec{r}, t) = \frac{q}{\sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}}{c \sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

solutions for
 \vec{r} in xy plane
 when charge passes
 through xy plane
 at $t=0$



at x
potential from charge at $vt \hat{z}$

potential at pt \vec{r} in xy plane
at time t , when charge is at
 $\vec{r}_0 = vt \hat{z}$, looks almost like
static Coulomb potential, which
would be $\frac{q}{\sqrt{r^2 + v^2 t^2}}$

But instead, it is

$$\frac{q}{\sqrt{v^2 t^2 + \left(\frac{r}{\gamma}\right)^2}}$$

looks like the transverse direction has contracted
by a factor γ !

Such considerations led Lorentz to discover
the Lorentz transformation, before Einstein
proposed his theory of special relativity

Radiation from a Localized Oscillating Source

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int d^3r' dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \vec{j}(\vec{r}', t')$$

For pure harmonic oscillation in current

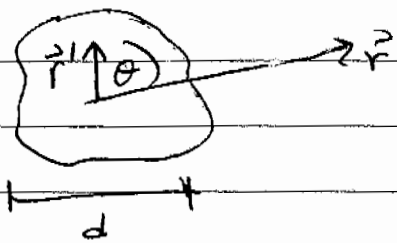
$$\vec{j}(\vec{r}, t) = \text{Re} \left\{ \vec{j}_\omega(\vec{r}) e^{-i\omega t} \right\}$$

$$\Rightarrow \vec{A}(\vec{r}, t) = \text{Re} \left\{ \vec{A}_\omega(\vec{r}) e^{-i\omega t} \right\}$$

$$\Rightarrow \vec{A}_\omega(\vec{r}) e^{-i\omega t} = \frac{1}{c} \int d^3r' \vec{j}_\omega(\vec{r}') e^{-i\omega t} \frac{e^{i\omega |\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c} \int d^3r' \vec{j}_\omega(\vec{r}') \frac{e^{i\omega |\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume source is localized, i.e. $\vec{j}_\omega(\vec{r}) \approx 0$ for $|\vec{r}| > d$



Approx ①

for $r \gg d$, far from sources

$$\begin{aligned} |\vec{r}-\vec{r}'| &= \sqrt{r^2 + r'^2 - 2rr' \cos \theta} \\ &= r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r} \cos \theta} \\ &\approx r \left(1 - \frac{r'}{r} \cos \theta\right) \\ &\approx r - \vec{r}' \cdot \hat{r} + o\left(\left(\frac{r'}{r}\right)^2\right) \end{aligned}$$

$\hat{r} \equiv \frac{\vec{r}}{r}$

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c} \int d^3r' \frac{\vec{j}_\omega(\vec{r}') e^{ik(r-\vec{r}'\cdot\hat{r})}}{r-\vec{r}'\cdot\hat{r}} \quad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{e^{ikr}}{cr} \int d^3r' \frac{\vec{j}_\omega(\vec{r}') e^{-ik\vec{r}'\cdot\hat{r}}}{1 - \frac{\hat{r}\cdot\vec{r}'}{r}}$$

$$\approx \frac{e^{ikr}}{cr} \int d^3r' \vec{j}_\omega(\vec{r}') e^{-ik\hat{r}\cdot\vec{r}'} \left(1 + \frac{\hat{r}\cdot\vec{r}'}{r}\right)$$

when combine with the $e^{-i\omega t}$ piece, this gives outgoing spherical wave $\frac{e^{i(kr-\omega t)}}{r}$

oscillating charge radiates outgoing spherical electromagnetic waves

the $\int d^3r' \vec{j}_\omega(\vec{r}')$ term will determine the angular dependence of the radiation.

Approx. ② $\lambda \gg d$ long wave length approx
or $kd \ll 1 \rightarrow \frac{\omega}{c} d \ll 1$ or $\frac{d}{\tau} \ll c$
where τ is period of oscillation.

Since $\frac{d}{\tau}$ is max speed of the oscillating charges $\rightarrow \lambda \gg d$ is a non-relativistic approximation

$kd \ll 1 \Rightarrow e^{-ik\hat{r} \cdot \vec{r}'} \approx 1 - ik\hat{r} \cdot \vec{r}' + \text{higher orders}$

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{cr} \int d^3r' \vec{j}_\omega(\vec{r}') (1 - ik\hat{r} \cdot \vec{r}') (1 + \frac{\hat{r} \cdot \vec{r}'}{r})$$

$$= \frac{e^{ikr}}{cr} \int d^3r' \vec{j}_\omega(\vec{r}') \left[1 + \hat{r} \cdot \vec{r}' \left(\frac{1}{r} - ik \right) \right]$$

+ higher order in $\frac{d}{r}$ or kd

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{r} \left[-\vec{I}_1 + \left(\frac{1}{r} - ik \right) \vec{I}_2 \right]$$

where $\vec{I}_1 \equiv \frac{1}{c} \int d^3r' \vec{j}_\omega(\vec{r}')$

$$\vec{I}_2 \equiv \frac{1}{c} \int d^3r' \hat{r} \cdot \vec{r}' \vec{j}_\omega(\vec{r}')$$

Consider first \vec{I}_1 i th component (\vec{I}_1 vanishes in statics)

$$\int d^3r j_i(\vec{r}) = - \int d^3r r_i \vec{\nabla} \cdot \vec{j} \quad \text{integration by parts}$$

$$= \int d^3r r_i \frac{\partial \rho}{\partial t} \quad \text{as } \vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

$$\int d^3r j_i(\vec{r}) = -i\omega \int d^3r r_i \rho_\omega(\vec{r})$$

$$\Rightarrow \vec{I}_1 = -\frac{i\omega}{c} \int d^3r \vec{r} \rho_\omega(\vec{r}) = -\frac{i\omega}{c} \vec{P}_\omega$$

electric dipole moment