Electric dipole approximation from $\vec{I}_1$

$$\boxed{\vec{A}_{E1}(\vec{r}) = \frac{e^{ikr}}{r}\left(\frac{-i\omega\vec{p}_\omega}{c}\right) = -i\,\vec{p}_\omega\,\frac{k\,e^{ikr}}{r}}$$ $\omega = ck$

Consider $\vec{I}_2$

$$\vec{I}_2 = \frac{1}{c}\int d^3r'\ \hat{r}\cdot\vec{r}'\ \vec{J}_\omega(\vec{r}') = \frac{1}{c}\hat{r}\cdot\underbrace{\int d^3r'\ \vec{r}'\ \vec{J}_\omega(\vec{r}')}_{\text{tensor}}$$

we saw this tensor earlier when we did the magnetic dipole approx, and when we derived the macroscopic Maxwell equations

$$\int d^3\vec{r}'\ \vec{r}'\vec{J}_\omega(\vec{r}') = -\int d^3r'\ \vec{J}_\omega(\vec{r}')\,\vec{r}' - \int d^3r'\ \vec{r}'\vec{r}'(\vec{\nabla}'\cdot\vec{J}_\omega(r'))$$

$$= \frac{1}{2}\int d^3r'\left[\vec{r}'\vec{J}_\omega - \vec{J}_\omega\vec{r}'\right] - \frac{1}{2}\int d^3r'\ i\omega\ \vec{r}'\vec{r}'\rho_\omega$$

using $\vec{\nabla}'\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$

$$\vec{I}_2 = \frac{1}{2c}\int d^3r'\left[(\hat{r}\cdot\vec{r}')\vec{J}_\omega - (\hat{r}\cdot\vec{J}_\omega)\vec{r}'\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'(\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\frac{1}{2c}\int d^3r'\left[\hat{r}\times(\vec{r}'\times\vec{J}_\omega)\right] - \frac{1}{2}\frac{i\omega}{c}\hat{r}\cdot\int d^3r'\ (\vec{r}'\vec{r}')\rho_\omega(\vec{r}')$$

$$= -\hat{r}\times\vec{m}_\omega - \frac{1}{2}\frac{i\omega}{3c}\hat{r}\cdot\overleftrightarrow{Q}'_\omega$$

where $\vec{m}_\omega = \frac{1}{2c}\int d^3r'\ \vec{r}'\times\vec{J}_\omega(\vec{r}')$ is magnetic dipole moment

$\overleftrightarrow{Q}'_\omega = \int d^3r'\ 3\vec{r}'\vec{r}'\ \rho_\omega(\vec{r}')$ looks almost like electric quadrapole tensor to make it look like the proper quadrapole moment

$$\overleftrightarrow{Q}_\omega = \int d^3r' \left(3\vec{r}'\vec{r}' - r'^2 \overleftrightarrow{I}\right) \rho_\omega(\vec{r}')$$

we can write

$$\overleftrightarrow{Q}'_\omega = \overleftrightarrow{Q}_\omega + \overleftrightarrow{I} \int d^3r' \, r'^2 \rho_\omega(\vec{r}')$$

$\uparrow$ identity matrix $I_{ij} = \delta_{ij}$

$$\overleftrightarrow{I}_2 = -\hat{r} \times \vec{m}_\omega - \frac{1}{2} \frac{i\omega}{3c} \hat{r} \cdot \overleftrightarrow{Q}_\omega - \frac{i\omega}{6c} \hat{r} \, C(\omega)$$

where $C_\omega \equiv \int d^3r' \, r'^2 \rho_\omega(\vec{r}')$ is a scalar

<u>Magnetic dipole approximation from $\overleftrightarrow{I}_2$</u>

$$\boxed{\vec{A}_{M1}(\vec{r}) = \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \left(-\hat{r} \times \vec{m}_\omega\right)}$$

<u>Electric quadrapole approximation from $\overleftrightarrow{I}_2$</u>

$$\boxed{\vec{A}_{E2}(\vec{r}) = \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik\right) \left(-\frac{i\omega}{6c} \hat{r} \cdot \overleftrightarrow{Q}_\omega\right)}$$

The last piece $\frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(\frac{-i\omega}{6c} \hat{r} \, C_\omega\right)$

can always be ignored - it is a radial function and so its curl always vanishes $\rightarrow$ gives no contribution to $\vec{B}$. Similarly, since $-\frac{i\omega}{c} \vec{E}_\omega = i\vec{k} \times \vec{B}_\omega$ by Ampere's law, this term will give no contribution to $\vec{E}$.

$\uparrow$ holds away from source where $\vec{J} = 0$.

So with these two approximations ① and ②

$$\vec{A}_\omega(\vec{r}) = \vec{A}_{E1}(\vec{r}) + \vec{A}_{M1}(\vec{r}) + \vec{A}_{E2}(\vec{r})$$

keeping higher order terms would give magnetic quadrupole, electric octopole etc.

Compare strengths of the terms above

Approx ③ Radiation zone: far from sources, $\frac{1}{r} \ll k$ so $\left(\frac{1}{r} - ik\right) \approx -ik$ in $\vec{A}_{M1}$ and $\vec{A}_{E2}$

only keep terms of $O\left(\frac{1}{r}\right)$

electric dipole $\quad \vec{p}_\omega \sim q d \qquad \vec{A}_{E1} \sim q k d$

magnetic dipole $\quad \vec{m}_\omega \sim \frac{v q d}{c} \qquad \vec{A}_{M1} \sim q k d \left(\frac{v}{c}\right)$

use $v \sim \frac{d}{t} \sim d\omega \sim dkc \Rightarrow \vec{A}_{M1} \sim q (kd)^2$

electric quadrupole $\quad \overleftrightarrow{Q}_\omega \sim q d^2 \qquad \vec{A}_{E2} \sim q d^2 k \frac{\omega}{c} \sim q (kd)^2$

Since Approx ② assumed $kd \approx \frac{v}{c} \ll 1$

above is expansion in powers of kd

leading term is electric dipole

next order are $\left[\begin{array}{l}\text{magnetic dipole} \\ \text{electric quadrupole}\end{array}\right] \qquad \frac{A_{M1}}{A_{E1}} \sim \frac{A_{E2}}{A_{E1}} \sim kd$

next order terms are smaller than A_{E1} by factor $(kd)^2$ etc.

Electric dipole Approximation — the leading term in non-relativistic expansion

$$\vec{A}_{E1}(\vec{r}) = -ik\vec{p}_\omega \frac{e^{ikr}}{r} \qquad \vec{\nabla}\times(\varphi\vec{F}) = (\vec{\nabla}\varphi)\times\vec{F} + \varphi\vec{\nabla}\times\vec{F}$$

$$\vec{B}_{E1} = \vec{\nabla}\times\vec{A}_{E1} = -ik\left(\vec{\nabla}\frac{e^{ikr}}{r}\right)\times\vec{p}_\omega$$

$$= -ik\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}_\omega$$

$$= k^2\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\hat{r}\times\vec{p}_\omega$$

In radiation zone approx, $kr \gg 1$

$$\boxed{\vec{B}_{E1} \simeq k^2\frac{e^{ikr}}{r}\,\hat{r}\times\vec{p}_\omega}$$

To get electric field, use Ampere's Law

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \qquad \text{(away from source where } \vec{J}=0\text{)}$$

For oscillating fields $\vec{E} = E_\omega e^{-i\omega t}$

$$\vec{\nabla}\times\vec{B}_\omega = -\frac{i\omega}{c}\vec{E}_\omega \Rightarrow E_{E1} = \frac{i}{k}\vec{\nabla}\times\vec{B}_{E1}$$

$$\vec{E}_{E1} = \frac{i}{k}\vec{\nabla}\times\left[k^2\frac{e^{ikr}}{r}\left(1+\frac{i}{kr}\right)\hat{r}\times\vec{p}_\omega\right]$$

ignore in RZ approx

$$\vec{E}_{E1} = \frac{i}{k}\left(\vec{\nabla} e^{ikr}\right) \times \left[\frac{k^2}{r}\left(1 + \frac{i}{kr}\right) \hat{r} \times \vec{p}_\omega \right]$$

$$+ \frac{i}{k} e^{ikr} \underbrace{\vec{\nabla} \times \left[\frac{k^2}{r}\left(1 + \frac{i}{kr}\right) \hat{r} \times \vec{p}_\omega \right]}$$

this will always be of order $1/r^2$
so can ignore it in radiation zone approx

So in <u>radiation zone approx</u>

$$\vec{E}_{E1} = \left(\vec{\nabla} e^{ikr}\right) \times \left[\frac{ik}{r} \hat{r} \times \vec{p}_\omega \right]$$

$$\boxed{\vec{E}_{E1} = -k^2 \frac{e^{ikr}}{r} \hat{r} \times \left(\hat{r} \times \vec{p}_\omega\right)}$$

- if do not make radiation zone approx, one gets

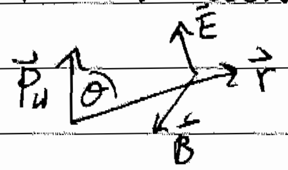
$$\vec{E}_{E1} = \frac{k^2 e^{ikr}}{r}\left[\vec{p}_\omega - \hat{r}(\vec{p}_\omega\cdot\hat{r}) - \frac{i}{kr}\left(1+\frac{i}{kr}\right)\left(3\hat{r}(\vec{p}_\omega\cdot\hat{r}) - \vec{p}\right)\right]$$

Using radiation zone approx:

$$\vec{E}_{E1} = k^2 \frac{e^{ikr}}{r}\,\hat{r}\times(\vec{p}_\omega\times\hat{r}) \qquad |\vec{E}_{E1}| = |\vec{B}_{E1}|$$

$$\vec{B}_{E1} = -k^2 \frac{e^{ikr}}{r}\,\vec{p}_\omega\times\hat{r} \qquad \vec{E}_{E1} \perp \vec{B}_{E1}$$

If choose coordinates so that $\vec{p}_\omega$ is along $\hat{z}$ axis, then



$$\vec{E}_{E1} = -k^2 p_\omega \frac{e^{ikr}}{r}\sin\theta\,\hat{\theta}$$

$$\vec{B}_{E1} = -k^2 p_\omega \frac{e^{ikr}}{r}\sin\theta\,\hat{\varphi}$$

Emitted power

Poynting vector $\vec{S}_{E1}(\vec{r},t) = \frac{c}{4\pi}\,\mathrm{Re}\{\vec{E}_{E1}\}\times\mathrm{Re}\{\vec{B}_{E1}\}$

need to take real parts of complex expression before multiplying

$$\mathrm{Re}\{\vec{E}_{E1}(\vec{r},t)\} = -k^2 p_\omega \frac{\cos(kr-\omega t)}{r}\sin\theta\,\hat{\theta}$$

$$\mathrm{Re}\{\vec{B}_{E1}(\vec{r},t)\} = -k^2 p_\omega \frac{\cos(kr-\omega t)}{r}\sin\theta\,\hat{\varphi}$$

$$\boxed{\vec{S}_{E1}(\vec{r},t) = \frac{c}{4\pi}k^4 p_\omega^2 \frac{\cos^2(kr-\omega t)}{r^2}\sin^2\theta\,\hat{r}}$$

$\vec{S}_{E1} \sim \hat{r} \Rightarrow$ energy flows radially outwards

$\vec{S}_{E1} \sim \frac{1}{r^2} \Rightarrow$ energy conserved

$$\oint_{\substack{\text{sphere}\\ \text{radius } R}} da\, \hat{n} \cdot \langle \vec{S}_{E1} \rangle = \text{constant for all } R$$

— Question - what about the $\frac{1}{r^n}$, $n \geq 2$, terms if we do not make radiation zone approx?

time averaged energy current

$$\langle \vec{S}_{E1} \rangle = \frac{1}{\tau} \int_0^{\tau} dt\, \vec{S}_{E1}(\vec{r}, t) \qquad \tau \text{ is period } \tau = \frac{2\pi}{\omega}$$

$$\langle \cos^2(\,) \rangle = \frac{1}{2}$$

$$= \frac{c}{8\pi} k^4 p_\omega^2 \frac{\sin^2\theta}{r^2} \hat{r}$$

average energy flowing through an element of area at spherical angles θ, φ is

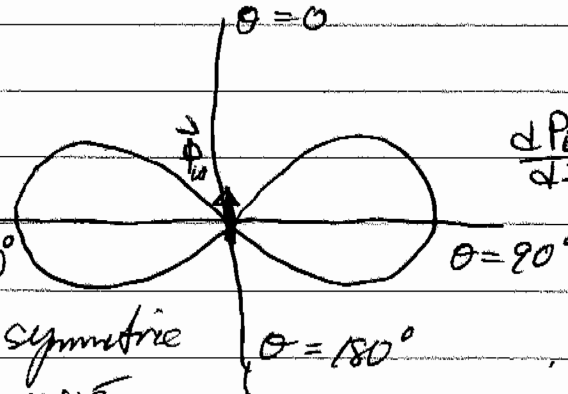
$$dP_{E1} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle \underbrace{r^2 \sin\theta\, d\theta\, d\varphi}_{\substack{\text{area of surface element}\\ = r^2 d\Omega}}$$

Ω is solid angle

$$= \hat{r} \cdot \langle \vec{S}_{E1} \rangle r^2 d\Omega$$

$$\frac{dP_{E1}}{d\Omega} = \hat{r} \cdot \langle \vec{S}_{E1} \rangle r^2 = \frac{c}{8\pi} k^4 p_\omega^2 \sin^2\theta \sim \omega^4 \sin^2\theta$$

polar plot



$\frac{dP_{E1}}{d\Omega} \sim \sin^2\theta$

most of power is directed outwards into plane $\perp \vec{p}_\omega$, ie peaked about $\theta = 90°$

rotationally symmetric about $\hat{z}$ axis

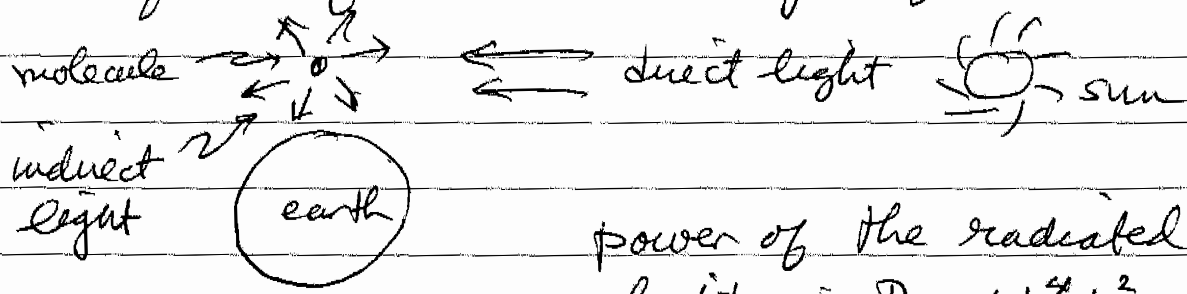
Total power radiated is

$$P_{E1} = \int \frac{dP_{E1}}{d\Omega} d\Omega = \frac{ck^4 p_\omega^2}{8\pi} \, 2\pi \underbrace{\int_0^\pi d\theta \sin\theta \sin^2\theta}_{4/3}$$

$$P_{E1} = \frac{ck^4 p_\omega^2}{3} = \frac{p_\omega^2 \omega^4}{3c^3} \sim \omega^4$$

<u>why the sky is blue – Lord Rayleigh</u>

When look up at sky, you are seeing the <u>indirect</u> light of the sun, which is the light emitted by the atoms and molecules of the atmosphere as they oscillate and so radiate, due to the electric field of the direct light from the sun



power of the radiated indirect light is $P \sim \omega^4 p_\omega^2$

$$\vec{p} = \alpha \vec{E} \qquad \alpha \sim \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

For molecules in atmosphere (N_2 etc) ω_0 is typically at a freq <u>higher</u> than visible spectrum. Therefore, in visible spectrum $\alpha \sim \frac{e^2}{m\omega_0^2}$ indep of ω.

$\Rightarrow$ power radiated is $P \sim \omega^4$

$P \sim \omega^4$ largest at high freq

Since light from sun is "white light"
it has components of all freqs. Of these
freqs, the higher ones are scattered the
most & make up the indirect light we see.

Since blue is the largest ω in visible spectrum,
the sky is blue!

When we look at sunrise or sunset, we
are looking at the direct rays of the sun.
Since these rays are least scattered at
low ω $\Rightarrow$ sunset and sunrise are red!

Magnetic dipole approx – Radiation Zone $kr \gg 1$

$$\vec{A}_{M1} = \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\hat{r}\times\vec{m}_\omega\right)$$

$$\simeq ik\,\hat{r}\times\vec{m}_\omega\,\frac{e^{ikr}}{r} \quad \text{in RZ}$$

$$\vec{B}_{M1} = \vec{\nabla}\times\vec{A}_{M1} = \left(\vec{\nabla}e^{ikr}\right)\times\left(\frac{ik\hat{r}\times\vec{m}_\omega}{r}\right)$$

$$+\, e^{ikr}\,\underbrace{\vec{\nabla}\times\left(\frac{ik\hat{r}\times\vec{m}_\omega}{r}\right)}$$

will give terms of $o\left(\frac{1}{r^2}\right)$
so ignore in RZ approx

$$\boxed{\vec{B}_{M1} = -k^2\frac{e^{ikr}}{r}\,\hat{r}\times\left(\hat{r}\times\vec{m}_\omega\right)}$$

From Ampere's Law

$$\vec{E}_{M1} = \frac{i}{k}\vec{\nabla}\times\vec{B}_{M1} = -ik\left(\vec{\nabla}e^{ikr}\right)\times\left(\frac{\hat{r}\times[\hat{r}\times\vec{m}_\omega]}{r}\right)$$

$$-\, ik\,e^{ikr}\,\underbrace{\vec{\nabla}\times\left(\frac{\hat{r}\times[\hat{r}\times\vec{m}_\omega]}{r}\right)}$$

will give terms of $o\left(\frac{1}{r^2}\right)$
so ignore in RZ approx

$$\vec{E}_{M1} = k^2\frac{e^{ikr}}{r}\,\hat{r}\times\left(\hat{r}\times\left(\hat{r}\times\vec{m}_\omega\right)\right)$$

triple product rule

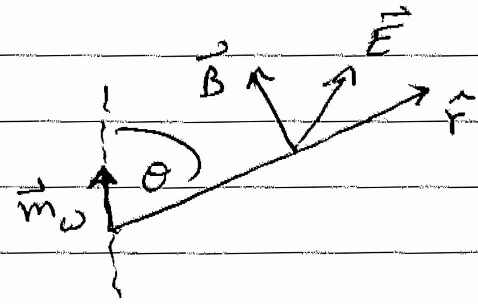
$$= k^2\frac{e^{ikr}}{r}\left\{\hat{r}\,[\hat{r}\cdot(\hat{r}\times\vec{m}_\omega)] - (\hat{r}\times\vec{m}_\omega)[\hat{r}\cdot\hat{r}]\right\}$$

$$\boxed{\vec{E}_{M1} = -\frac{k^2}{r}e^{ikr}\left(\hat{r}\times\vec{m}_\omega\right)}$$

If allign axes so that $\vec{m}_\omega = m_\omega \hat{z}$ then

$$\vec{E}_{M1} = m_\omega \frac{k^2}{r} e^{ikr} \sin\theta\, \hat{\varphi}$$

$$\vec{B}_{M1} = -m_\omega \frac{k^2}{r} e^{ikr} \sin\theta\, \hat{\theta}$$



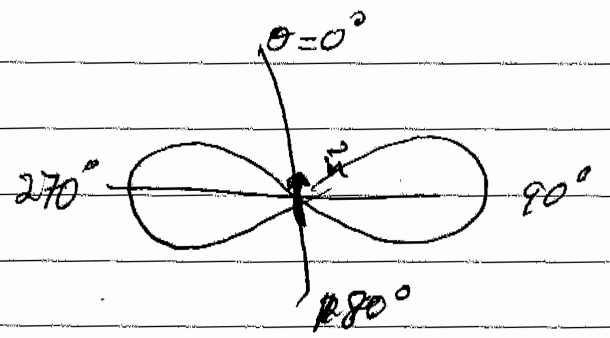
<u>Poynting vector</u>

$$\vec{S}_{M1} = \frac{c}{4\pi} Re\{\vec{E}_{M1}\} \times Re\{\vec{B}_{M1}\}$$

$$= \frac{c}{4\pi} \frac{k^4 m_\omega^2}{r^2} \cos^2(kr-\omega t) \sin^2\theta\ \hat{r}$$

$$\langle \vec{S}_{M1} \rangle = \frac{c}{8\pi} k^4 m_\omega^2 \frac{\sin^2\theta}{r^2}\ \hat{r}$$

$$\frac{dP_{M1}}{d\Omega} = \hat{r}\cdot\langle \vec{S}_{M1}\rangle r^2 = \frac{c}{8\pi} k^4 m_\omega^2 \sin^2\theta \sim \omega^4 \sin^2\theta$$



rotationally symmetric about $\hat{z}$ axis

$$P_{M1} = \int d\Omega \frac{dP_{M1}}{d\Omega} = \frac{ck^4}{3} m_\omega^2 = \frac{m_\omega^2 \omega^4}{3c^3}$$

$$\frac{P_{M1}}{P_{E1}} = \frac{m_\omega^2}{p_\omega^2} \sim \left(\frac{v}{c}\right)^2 \qquad m_\omega \sim \frac{dj}{c} \sim dq\frac{v}{c}$$

$$p_\omega \sim dq$$

Electric Quadrupole radiation – radiation zone approx

$$\vec{A}_{E2} = \frac{e^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(\frac{-i\omega}{6c}\,\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)$$

$$= -\frac{e^{ikr}}{r}\,\frac{k^3}{6}\,\hat{r}\cdot\overleftrightarrow{Q}_\omega \quad \text{in RZ approx}$$

$$\vec{B}_{E2} = \vec{\nabla}\times\vec{A}_{E2} = -(\vec{\nabla}e^{ikr})\times\left[\frac{k^2\,\hat{r}\cdot\overleftrightarrow{Q}_\omega}{6r}\right]$$

$$- e^{ikr}\,\underbrace{\vec{\nabla}\times\left[\frac{k^2\,\hat{r}\cdot\overleftrightarrow{Q}_\omega}{6r}\right]}_{O\left(\frac{1}{r^2}\right)\text{ so ignore in RZ approx.}}$$

$$\boxed{\vec{B}_{E2} = -ik^3\,\frac{e^{ikr}}{6r}\,\hat{r}\times\left(\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)}$$

$$\vec{E}_{E2} = \frac{i}{k}\vec{\nabla}\times\vec{B}_{E2} = k^2(\vec{\nabla}e^{ikr})\times\left[\frac{\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q}_\omega)}{6r}\right]$$

$$+ k^2 e^{ikr}\,\underbrace{\vec{\nabla}\times\left[\frac{\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q}_\omega)}{6r}\right]}_{O\left(\frac{1}{r^2}\right)\text{ so ignore in RZ approx}}$$

$$\vec{E}_{E2} = ik^3\,\frac{e^{ikr}}{6r}\,\underbrace{\hat{r}\times\left[\hat{r}\times(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right]}_{\text{triple product rule}}$$

$$= \frac{ik^3e^{ikr}}{6r}\left\{\hat{r}\left[\hat{r}\cdot(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right] - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)\left[\hat{r}\cdot\hat{r}\right]\right\}$$

$$\boxed{\vec{E}_{E2} = \frac{ik^3e^{ikr}}{6r}\left\{\hat{r}\left(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r}\right) - \left(\hat{r}\cdot\overleftrightarrow{Q}_\omega\right)\right\}}$$

Poynting vector

$$\vec{S}_{E2} = +\frac{c}{4\pi}\, Re\{\vec{E}_{E2}\} \times Re\{\vec{B}_{E2}\}$$

$$= \frac{(c/4\pi)\,-k^6}{36 r^2}\sin^2(kr-\omega t)\left\{\hat{r}(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r}) - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right\}\times\left[\hat{r}\times(\hat{r}\cdot\right.$$

$$= \frac{(c/4\pi)\,-k^6}{36 r^2}\sin^2(kr-\omega t)\Big\{\hat{r}\left[\hat{r}(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})\cdot(\hat{r}\,\overleftrightarrow{Q}_\omega)\right] - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)\left[\hat{r}(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})\cdot\hat{r}\right]$$

$$- \hat{r}\left[(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\cdot(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\right] + (\hat{r}\cdot\overleftrightarrow{Q}_\omega)\left[(\hat{r}\cdot\overleftrightarrow{Q}_\omega)\cdot\hat{r}\right]\Big\}$$

$$= \frac{(c/4\pi)\,-k^6}{36 r^2}\sin^2(kr-\omega t)\Big\{(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2\hat{r} - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})$$

$$-(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})\hat{r}\;(\hat{r}\cdot\overleftrightarrow{Q}_\omega)(\hat{r}\cdot\overleftrightarrow{Q}_\omega$$

$$\vec{S}_{E2} = \frac{-ck^6}{4\pi\, 36 r^2}\sin^2(kr-\omega t)\left\{(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2 - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2\right\}\hat{r}$$

$$\langle\vec{S}_{E2}\rangle = \frac{-ck^6}{4\pi\, 72 r^2}\left\{(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2 - (\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2\right\}\hat{r}$$

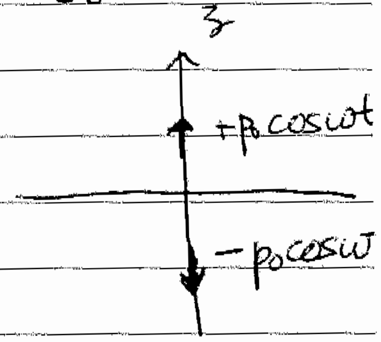
$$\frac{dP_{E2}}{d\Omega} = \hat{r}\cdot\langle\vec{S}_{E2}\rangle r^2 = \frac{ck^6}{4\pi\, 72}\left\{(\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 - (\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2\right\}$$

angular dependence of $\frac{dP_{E2}}{d\Omega}$ depends on specific form of the tensor $\overleftrightarrow{Q}_\omega$

For example: suppose $Q_{ij} = 0$ except for Q_{zz}

$$\Rightarrow \overleftrightarrow{Q}_\omega = Q_{zz}\,\hat{z}\hat{z}$$

ex



$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r})^2 = (Q_{zz}\cos^2\theta)^2$$

$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 = Q_{zz}^2\cos^2\theta$$

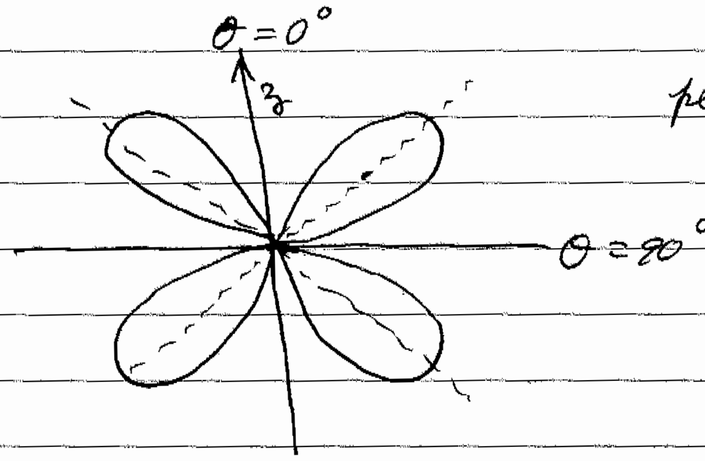
$$\frac{dP_{E2}}{d\Omega} = \frac{ck^6}{4\pi 72} Q_{zz}^2\left[\cos^2\theta - \cos^4\theta\right]$$

$$= \frac{ck^6}{4\pi 72} Q_{zz}^2 \cos^2\theta\sin^2\theta$$

$\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$

$$= \frac{ck^6}{4\pi 288} Q_z^{2\,2}\sin^2 2\theta$$

$\frac{dP_{E2}}{d\Omega}$



peak at 45°

rotationally invariant about $\hat{z}$ axis

$$\frac{P_{E2}}{P_{E1}} \sim \frac{k^6 Q^2}{k^4 p^2} \sim \frac{k^2 (qd^2)^2}{(qd)^2} \sim k^2 d^2 \sim \left(\frac{v}{c}\right)^2$$

$$P_{E2} \sim P_{M1}$$

For more general case, choose axes so that $\overleftrightarrow{Q}_\omega$ is diagonal - can always do this since $\overleftrightarrow{Q}_\omega$ is symmetric

$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega\cdot\hat{r}) = \hat{r}\cdot\begin{pmatrix} Q_{xx} & 0 & 0 \\ 0 & Q_{yy} & 0 \\ 0 & 0 & Q_{zz}\end{pmatrix}\cdot\hat{r}$$

$$= \hat{r}\cdot\begin{pmatrix} Q_{xx}\sin\theta\cos\varphi \\ Q_{yy}\sin\theta\sin\varphi \\ Q_{zz}\cos\theta\end{pmatrix} = \begin{aligned} &Q_{xx}\sin^2\theta\cos^2\varphi \\ &+ Q_{yy}\sin^2\theta\sin^2\varphi \\ &+ Q_{zz}\cos^2\theta\end{aligned}$$

$$(\hat{r}\cdot\overleftrightarrow{Q}_\omega)^2 = Q_{xx}^2\sin^2\theta\cos^2\varphi + Q_{yy}^2\sin^2\theta\sin^2\varphi + Q_{zz}^2\cos^2\theta$$

$$\frac{dP_{E2}}{d\Omega} = \frac{k^6}{72}\Big\{ Q_{zz}^2(\cos^2\theta-\cos^4\theta) + Q_{xx}^2(\sin^2\theta\cos^2\varphi - \sin^4\theta\cos^4\varphi) + Q_{yy}^2(\sin^2\theta\sin^2\varphi - \sin^4\theta\sin^4\varphi\Big\}$$

$$\frac{dP_{E2}}{d\Omega} = \frac{k^6}{72}\Big\{ Q_{zz}^2\cos^2\theta\sin^2\theta + Q_{xx}^2\sin^2\theta\cos^2\varphi\,(1-\sin^2\theta\cos^2\varphi) + Q_{yy}^2\sin^2\theta\sin^2\varphi\,(1-\sin^2\theta\sin^2\varphi)\Big\}$$

no special symmetries - varies with θ and φ

For arbitrary charge distributions – not pure harmonic

For $\vec{p}_\omega e^{-i\omega t}$ pure harmonic oscillation, we found the radiated fields in electric dipole approx are

$$\vec{E} = \vec{E}_\omega e^{-i\omega t}, \quad \vec{B} = \vec{B}_\omega e^{-i\omega t}$$

$$\vec{E}_\omega = -k^2 \frac{e^{ikr}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega) = -\frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$\vec{B}_\omega = k^2 \frac{e^{ikr}}{r} (\hat{r} \times \vec{p}_\omega) = \frac{\omega^2}{c^2} \frac{e^{i\omega r/c}}{r} (\hat{r} \times \vec{p}_\omega)$$

as $k = \frac{\omega}{c}$

For an arbitrarily time varying charge distribution with electric dipole moment

$$\vec{p}(t) = \int \frac{d\omega}{2\pi} \vec{p}_\omega e^{-i\omega t}$$

then solution for fields given by superposition

$$\vec{E}(\vec{r},t) = \int \frac{d\omega}{2\pi} \vec{E}_\omega e^{-i\omega t}$$

$$= -\int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-r/c)}}{r} \left(\frac{\omega^2}{c^2}\right) \hat{r} \times (\hat{r} \times \vec{p}_\omega)$$

$$= \frac{-1}{c^2 r} \hat{r} \times \left[\hat{r} \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega \, \omega^2\right]$$

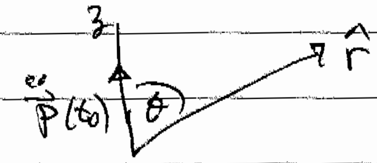
$$= \frac{1}{c^2 r} \hat{r} \times \left[\hat{r} \times \frac{\partial^2}{\partial t^2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \vec{p}_\omega\right]$$

$$\boxed{\vec{E}(\vec{r},t) = \frac{1}{c^2 r}\,\hat{r}\times\left[\hat{r}\times\ddot{\vec{p}}(t-r/c)\right]} \qquad \ddot{\vec{p}} = \frac{d^2\vec{p}}{dt^2}$$

define $t_0 \equiv t - r/c$ = "retarded time"

in spherical coords, if $\ddot{\vec{p}}(t_0)$ is along $\hat{z}$

$$\vec{E}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\,\hat{\theta}$$



Similarly

$$\vec{B}(\vec{r},t) = \int \frac{d\omega}{2\pi}\,\vec{B}_\omega\, e^{-i\omega t}$$

$$= \int \frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-r/c)}}{r}\left(\frac{\omega^2}{c^2}\right)\left(\hat{r}\times\vec{p}_\omega\right)$$

$$= \frac{-1}{c^2 r}\,\hat{r}\times\frac{\partial^2}{\partial t^2}\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-r/c)}\,\vec{p}_\omega$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{-1}{c^2 r}\,\hat{r}\times\ddot{\vec{p}}(t_0)}$$

$$\vec{B}(\vec{r},t) = \frac{\ddot{p}(t_0)}{c^2 r}\sin\theta\,\hat{\varphi} \qquad \text{in spherical coords}$$

<u>Poynting vector</u>

$$\vec{S} = \frac{c}{4\pi}\,\vec{E}\times\vec{B} = \frac{c}{4\pi}\left(\frac{1}{c^2 r}\right)^2\left[\ddot{p}(t_0)\right]^2\sin^2\theta\,\hat{r}$$

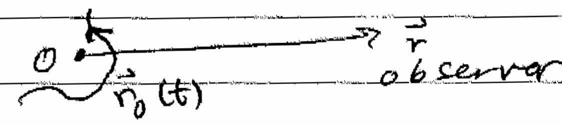
Total power radiated through a sphere of radius r is

$$\mathcal{P} = \oint da\, \hat{r} \cdot \vec{S} = 2\pi \int_0^{\pi} d\theta \sin\theta\, r^2\, \hat{r} \cdot \vec{S}$$

$$= \frac{[\ddot{p}(t_0)]^2}{2c^3} \underbrace{\int_0^{\pi} d\theta \sin^3\theta}_{4/3}$$

$$\boxed{\mathcal{P} = \frac{2\,[\ddot{p}(t_0)]^2}{3c^3}}$$

For a point charge moving along a trajectory $\vec{r}_0(t)$



$$\vec{p}(t) = q\,\vec{r}_0(t)$$

$$\ddot{\vec{p}}(t) = q\,\ddot{\vec{r}}_0(t) = q\,\vec{a}(t)$$

↑ acceleration

$$\boxed{\mathcal{P} = \frac{2}{3}\frac{q^2 a^2(t_0)}{c^3}}$$

← total power passing through a sphere of radius r at time t is due to acceleration at retarded time $t_0 = t - r/c$

Larmor's formula

power radiated ~ (acceleration)^2

Larmor's formula above only holds in the non-relativistic limit since it is based on the electric dipole approx.

To go beyond non-relativistic limit, one can do the following:

1) transform to a new inertial frame of reference in which the charge is <u>instantaneously</u> at rest
2) apply non-relativistic Larmor formula in this frame
3) transform back to "lab" frame

<u>Back to non-relativistic case</u>

- radiation fields from moving point charge

$$\vec{E}(\vec{r},t) = \frac{q}{c^2 r}\,\hat{r}\times(\hat{r}\times\vec{a}(t_0))$$

$$\vec{B}(\vec{r},t) = -\frac{q}{c^2 r}\,(\hat{r}\times\vec{a}(t_0))$$

<u>Poynting vector</u>

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = -\frac{c}{4\pi}\left(\frac{q}{c^2 r}\right)^2\left[\hat{r}\times(\hat{r}\times\vec{a})\right]\times\left[\hat{r}\times\vec{a}\right]$$

$$= \frac{c}{4\pi}\left(\frac{q}{c^2 r}\right)^2\left\{\hat{r}\,(\hat{r}\times\vec{a})^2 - (\hat{r}\times\vec{a})\left[\hat{r}\cdot(\hat{r}\times\vec{a})\right]\right.$$

$$= \frac{q^2}{4\pi c^3}\left[\frac{\hat{r}\times\vec{a}(t_0)}{r^2}\right]^2\hat{r}$$

$$\boxed{\begin{aligned}\vec{S} &= \frac{q^2}{4\pi c^3 r^2}\left(a^2 - (\hat{r}\cdot\vec{a})^2\right)\hat{r}\\ &= \frac{q^2}{4\pi c^3}\frac{a^2(t_0)}{r^2}\sin^2\theta\,\hat{r}\end{aligned}}$$

for $\vec{a} = a\hat{z}$

$\vec{a}$ evaluated at t_0