Special Relativity

1) Speed of light is constant in all inertial frames of reference.
2) Physical laws must look the same in all inertial frames of reference — there is no experiment that can determine the "absolute" velocity of any inertial frame.

⇒ If a flash of light goes off at the origin of some coord system, the outgoing wavefronts look spherical in all inertial frames.

Equation of wavefront: \( r^2 - c^2t^2 = 0 \)

⇒ \((x, y, z, t)\) coords in one inertial frame \(K\)

\((x', y', z', t')\) coords in another inertial frame \(K'\) that moves with velocity \(\vec{v} = v\hat{x}\) with respect to \(K\).

What is the transformation that relates coords in \(K\) to coords in \(K'\)?

\[ y = y', \quad z = z' \]

⇒ \( c^2t^2 - x^2 = c^2t'^2 - x'^2 \)

⇒ \( \frac{(ct+x)(ct-x)}{(ct'+x')(ct'-x')} = 1 \)

Expect transformation to be linear.

⇒ \( ct' + x' = (ct + x) f \)

\( ct' - x' = (ct - x) f^{-1} \)

for some constant \(f\). Write \(f = e^{-y}\) \(y\) is rapidity.
Solve for $c^t$ and $x'$ in terms of $c^t$ and $x$

$$c^t' = c^t \left( \frac{e^y + e^{-y}}{2} \right) - x \left( \frac{e^y - e^{-y}}{2} \right)$$

$$x' = -c^t \left( \frac{e^y - e^{-y}}{2} \right) + x \left( \frac{e^y + e^{-y}}{2} \right)$$

$$c^t' = c^t \cosh y - x \sinh y$$

$$x' = -c^t \sinh y + x \cosh y$$

meaning of parameter $y$

$$y = \tan^{-1}(\frac{c^t}{x})$$

The origin of $K'$ has trajectory $x' = -ct'$ in $K'$

$$\Rightarrow \frac{x'}{c^t'} = -v$$

From transformation above, with $x = 0$, we get

$$\frac{x'}{ct'} = \frac{-ct \sinh y}{ct \cosh y} = -\tanh y$$

$$\Rightarrow \frac{v}{c} = \tanh y$$

$$\Rightarrow \cosh y = \frac{1}{\sqrt{1-(\frac{v}{c})^2}} \equiv \gamma$$

$$\sinh y = \frac{v}{c} \gamma$$

Lorentz Transformation

$$\begin{cases} c^t' = \gamma c^t - \gamma \frac{v}{c} x \\ x' = -\gamma \frac{v}{c} c^t + \gamma x \end{cases}$$
Inverse transform obtained by taking \( u \rightarrow -u \) in above:

\[
\begin{align*}
ct' &= \gamma ct + \delta \left( \frac{v}{c} \right) x' \\
x' &= \gamma \left( \frac{v}{c} \right) ct' + \gamma x
\end{align*}
\]

\[
\gamma - \text{vectors}
\]

\[
\gamma - \text{position: } x_{\mu} = (x_1, x_2, x_3, ct) \quad x_4 = ct
\]

\[
x_{\mu} x_{\mu} = \sum_{\mu=1}^{4} x_{\mu}^2 = x^2 - c^2 t^2 \quad \text{lorentz invariant scalar}
\]

\[
\text{invariant in all internal frames}
\]

\[
x_1' = \gamma \left( x_1 + \frac{v}{c} x_4 \right)
\]

\[
x_2' = x_2
\]

\[
x_3' = x_3
\]

\[
x_4' = \gamma \left( x_4 - \frac{v}{c} x_1 \right)
\]

\[
\text{linear transfo, can be represented by a matrix}
\]

\[
\mathbf{L} = \begin{pmatrix}
\gamma & 0 & 0 & \frac{-v}{c} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{-v}{c} \gamma & 0 & 0 & \gamma
\end{pmatrix}
\]

\[
\text{inverse: } x_{\mu} = a_{\mu\nu}(L^{-1}) x_{\nu}'
\]

\[
\text{we see } a_{\mu\nu}(L^{-1}) = a_{\nu\mu}(L)
\]

\[
\text{inverse = transpose}
\]
More generally,

Since $x^{2}$ is Lorentz invariant scalar,

$$x_{\mu}^{2} = \alpha_{\mu}(\mathbf{L}) \alpha_{\lambda}(\mathbf{L}) x_{\nu} x_{\lambda} = x^{2}$$

$$\Rightarrow \alpha_{\mu}(\mathbf{L}) \alpha_{\lambda}(\mathbf{L}) = \delta_{\mu\lambda}$$

$$\Rightarrow \alpha_{\mu}(\mathbf{L}) \alpha^{-1}_{\mu}(\mathbf{L}) = \delta_{\nu\lambda}$$

$$\Rightarrow \alpha_{\mu} \cdot \alpha^{-1}_{\mu} = \delta_{\nu\lambda} \quad \text{transpose = inverse}$$

$\alpha$ is a 4x4 orthogonal matrix

If $\mathbf{L}_1$ is a Lorentz transform from $\mathbf{K}$ to $\mathbf{K}'$

$\mathbf{L}_2$ is a Lorentz transform from $\mathbf{K}'$ to $\mathbf{K}''$

Then the Lorentz transform from $\mathbf{K}$ to $\mathbf{K}''$ is given by the matrix

$$\alpha(\mathbf{L}_2 \mathbf{L}_1) = \alpha(\mathbf{L}_2) \alpha(\mathbf{L}_1)$$

$$\Rightarrow \alpha^{-1}(\mathbf{L}) = \alpha(\mathbf{L}^{-1})$$

$$dx_{\mu} = (dx_1, dx_2, dx_3, i \, cdt)$$

$$-(dx_{\mu})^2 = c^2 ds^2 = c^2 dt^2 - dr^2 \quad \text{Lorentz invariant scalar}$$

$$ds^2 = \left[ t^2 - \frac{1}{c^2} \left( \frac{dx_1}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_2}{dt} \right)^2 - \frac{1}{c^2} \left( \frac{dx_3}{dt} \right)^2 \right]$$

$$ds^2 = \frac{dt^2}{\gamma^2}$$

$$ds = \frac{dt}{\gamma} \quad \text{proper time interval}$$
A 4-vector is any 4 numbers that transform under a Lorentz transformation the same way as does $x_\mu$

\[ \text{4-velocity} \quad U_\mu = \frac{dx_\mu}{ds} = \gamma x_\mu \]

\[ = \gamma \frac{dx_\mu}{dt} \]

Space components $\ddot{u} = \gamma \ddot{v}$

$U_4 = ic\gamma$

\[ U_\mu U_\mu = \gamma^2 v_1^2 - c^2 \gamma^2 = \gamma^2 (v^2 - c^2) \]

\[ = \frac{v^2 - c^2}{1 - \frac{v^2}{c^2}} = -c^2 \]

4-acceleration $\alpha_\mu = \gamma \frac{dU_\mu}{dt}$

4-gradient $\nabla_\mu = \left( \frac{\partial}{\partial x_\mu}, -\frac{ic}{c} \frac{\partial}{\partial t} \right)$

Proof $\frac{\partial}{\partial x_\mu}$ is a 4-vector

\[ \frac{\partial}{\partial x_\mu} = \frac{\partial x_\lambda}{\partial x_\mu} \frac{\partial}{\partial x_\lambda} \quad \text{but} \quad \frac{\partial x_\lambda}{\partial x_\mu} = \delta_{\lambda \mu} (L^{-1}) \]

\[ = \delta_{\lambda \mu} (L) \frac{\partial}{\partial x_\lambda} \]

So transforms same as $x_\mu$

\[ \left( \frac{\partial}{\partial x_\mu} \right)^2 = \nabla_\mu^2 - \frac{i}{c^2} \frac{\partial^2}{\partial t^2} \]

wave equation operator!

\[ \text{main products} \]

If $U_\mu$ and $V_\mu$ are 4-vectors, then

$U_\mu V_\mu$ is Lorentz invariant scalar
Electromagnetism

Clearly \( \mathbf{E} + \mathbf{B} \) must transform with each other under Lorentz transit.

In material frame \( K \)

Stationary line charge \( \lambda \)

Cylindrical outward electric field

\( \mathbf{E} \times \mathbf{I} \) (outward radial)

Moving line charge gives current \( \Rightarrow \mathbf{B} \) circulating around wire as well as outward radial \( \mathbf{E} \)

Lorentz force

\[
\mathbf{F} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B}
\]

What is the velocity \( \mathbf{v} \) here? Velocity with respect to what material frame? Clearly \( \mathbf{E} \) and \( \mathbf{B} \) must change from material frame to another if this force law can make sense.

Charge density

Consider charge \( \Delta q \) contained in a vol \( \Delta V \). \( \Delta q \) is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame.
\[ \Delta Q = \hat{\rho} \Delta V \]
\( \hat{\rho} \) is charge density in the rest frame
\( \Delta V \) is volume in the rest frame

\( \hat{\rho} \) is Lorentz invariant by definition.

Now, transform to another frame moving with \( \vec{u} \) with respect to rest frame.

\( \Delta Q \) remains the same.

\( \Delta V = \Delta V / \gamma \) volume contracts in direction \( \vec{u} \) to \( \vec{v} \).

\( \hat{\rho} = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta V} \gamma = \hat{\rho} \gamma \)

Current density is \( \vec{j} = \hat{\rho} \vec{u} = \gamma \vec{v} \cdot \vec{p} = \hat{\rho} \gamma \vec{u} \)

Define 4-current \( \vec{j}_\mu = (\vec{j}, E, \gamma) = \hat{\rho} (\gamma \vec{u}, \vec{E}, \gamma) = \hat{\rho} \gamma \vec{u}_\mu \)

It is 4-vector since \( \vec{u}_\mu \) is 4-vector and \( \hat{\rho} \) is Lorentz invariant, scalar.

Charge conservation

\[ \nabla \cdot \vec{j} + \frac{\partial \hat{\rho}}{\partial t} = \frac{\partial j_\mu}{\partial x_\mu} = 0 \]
Equation for potentials in Lorentz gauge

\[ (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \vec{A} = -\frac{4\pi}{c} \vec{J} \]

\[ (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \phi = -\frac{4\pi}{c} \rho \]

\[ \frac{\partial^2}{\partial x_i^2} = (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \text{ is Lorentz invariant operator} \]

\[ A_{\mu} = (\vec{A}, i\phi) \]

\[ (\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) A_{\mu} = -\frac{4\pi}{c} j_{\mu} = \frac{\partial^2 A_{\mu}}{\partial x^2} \]

Lorentz gauge condition is:

\[ \nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = \frac{\partial A_{\mu}}{\partial x^\mu} = 0 \]

Electric and magnetic fields

\[ B_i = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \quad \text{\textit{\epsilon}_{ijk} cyclic permutation of } 1,2,3 \]

\[ E_i = -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{c \partial t} = \frac{1}{c} \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \]

Define field stress tensor

\[ F_{\mu \nu} = \frac{\partial A_{\nu}}{\partial x^\mu} - \frac{\partial A_{\mu}}{\partial x^\nu} = -F_{\nu \mu} \]

\[ = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ -iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \]

"Curl" of a 4-vector is a 4x4 antisymmetric 2nd rank tensor.
Inhomogeneous Maxwell's equations can be written in the form

\[ \frac{\partial F_{\mu\nu}}{\partial x^\nu} = \frac{4\pi}{c} \frac{\hat{f}_\mu}{x^2} \]

\[ \nabla \times \vec{E} = \frac{4\pi}{c} \vec{J} \]

\[ \nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi i}{c} \vec{J} \]

\[ \frac{\partial}{\partial x^\nu} \left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) - \frac{\partial}{\partial x^\mu} \left( \frac{\partial A_\mu}{\partial x^\nu} \right) \]

\[ = - \frac{\partial^2 A_\mu}{\partial x^\nu \partial x^\nu} = \frac{4\pi}{c} \frac{\hat{f}_\mu}{x^2} \]

agrees with previous equation for \( A_\mu \).

Transformation law for 2nd rank tensor \( F_{\mu\nu} \)

\[ F'_{\mu\nu} = \frac{\partial x^\nu}{\partial x'^\nu} - \frac{\partial x^\mu}{\partial x'^\nu} \]

use \( A'_\mu = A_\mu \sigma A_\sigma \)

\[ \frac{\partial A_\mu}{\partial x'^\nu} = \frac{\partial A_\mu}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu} \]

\[ = \sigma_{\mu\sigma} \frac{\partial A_\sigma}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu} \]

\[ = \sigma_{\mu\sigma} a_{\nu\sigma} \frac{\partial A_\sigma}{\partial x^\rho} \frac{\partial x^\rho}{\partial x'^\nu} \]

\[ F'_{\mu\nu} = \sigma_{\mu\sigma} a_{\nu\sigma} F_{\sigma\rho} \]

\[ \text{gets one finds } \vec{E}' \text{ and } \vec{B}' \]

For \( n \) th rank tensor \( T_{\mu_1 \mu_2 ... \mu_n} \) of one knows \( \vec{E} \) and \( \vec{B} \)

\[ T'_{\mu_1 \mu_2 ... \mu_n} = a_{\mu_1} \sigma a_{\mu_2} \sigma ... a_{\mu_n} \sigma T_{\nu_1 \nu_2 ... \nu_n} \]
\[ \frac{\partial F_{\mu \nu}}{\partial x^\lambda} = \text{a 4-vector: proof} \]

\[ \frac{\partial F_{\mu \nu}}{\partial x^\lambda} = \alpha_\mu \alpha_\nu \alpha_\lambda \frac{\partial F_{\sigma \lambda}}{\partial x^\gamma} \]

but \( \alpha_\nu = \alpha^\nu \) since inverse = transpose

\[ \alpha_\nu \alpha_\lambda \alpha_\gamma = \alpha^\nu \alpha^\lambda \alpha^\gamma = 8 \times 8 \]

\[ \frac{\partial F_{\mu \nu}}{\partial x^\lambda} = \alpha_\mu \frac{\partial F_{\sigma \lambda}}{\partial x^\gamma} \quad \text{transforms like 4-vector} \]

To write the homogeneous Maxwell Equations

Construct 3rd rank co-variant tensor

\[
G_{\mu \nu \lambda} = \frac{\partial F_{\mu \nu}}{\partial x^\lambda} + \frac{\partial F_{\nu \lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda \mu}}{\partial x^\nu}
\]

transforms as \( G_{\mu \nu \lambda} = \alpha_\mu \alpha_\nu \alpha_\lambda G_{\mu \nu \lambda} \)

in principle \( G \) has \( 4^3 = 64 \) components

But can show that \( G \) is antisymmetric in exchange of any two indices

\[
G_{\mu \nu \lambda} = \frac{\partial F_{\mu \nu}}{\partial x^\lambda} + \frac{\partial F_{\nu \lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda \mu}}{\partial x^\nu}
\]

\[ = - \frac{\partial F_{\mu \nu}}{\partial x^\lambda} - \frac{\partial F_{\nu \lambda}}{\partial x^\mu} - \frac{\partial F_{\lambda \mu}}{\partial x^\nu} \quad \text{as } F \text{ anti-symmetric} \]

\[ = - G_{\nu \mu \lambda} \]
Also \( G_{\mu\nu\lambda} = 0 \) if any two indices are equal

\( \Rightarrow \) only 4 independent components

\( G_{012}, G_{013}, G_{023}, G_{123} \)

all other components either vanish or are \pm one of the above.

The 4 homogeneous Maxwell Equations:

\[ \nabla \cdot B = 0, \quad \nabla \times E + \frac{1}{c^2} \frac{\partial B}{\partial t} = 0 \]

can be written as

\[ G_{\mu\nu\lambda} = 0 \]

to see, substitute in definition of \( G \) the definition of \( F \)

\[ G_{\mu\nu\lambda} = \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} - \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} - \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} + \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda} \]

all terms cancel in pairs

\[ = 0 \]

\[ G_{123} = 0 \Rightarrow \nabla \cdot B = 0 \]

\[ G_{012} = -\omega \left[ \nabla \times E + \frac{\partial B}{\partial t} \right] = 0 \quad \text{3 component Faraday's Law} \]
Another way to write homogeneous Maxwell Equations

Define \( \epsilon_{\mu \nu \lambda \sigma} = \begin{cases} +1 & \text{if } \mu \nu \lambda \sigma \text{ is even permutation of 1234} \\ -1 & \text{if } \mu \nu \lambda \sigma \text{ is odd permutation of 1234} \\ 0 & \text{otherwise} \end{cases} \)

4-d Levi-Civita symbol

Define \( \tilde{F}_{\mu \nu} = \frac{i}{2} \epsilon_{\mu \nu \lambda \sigma} F_{\lambda \sigma} \) pseudo-tensor

\[
\tilde{F}_{\mu \nu} = \begin{pmatrix}
0 & -E_3 & E_2 & -iB_1 \\
E_3 & 0 & -E_1 & -iB_2 \\
-E_2 & E_1 & 0 & -iB_3 \\
iB_1 & iB_2 & iB_3 & 0
\end{pmatrix}
\]

\( \tilde{F}_{\mu \nu} = 0 \) gives homogeneous Maxwell equations

\( \frac{\partial \tilde{F}_{\mu \nu}}{\partial x_\nu} = 0 \)

\( \frac{1}{2} \tilde{F}_{\mu \nu} F_{\mu \nu} = B^2 - E^2 \) Lorentz invariant scalars

\( -\frac{1}{4} F_{\mu \nu} \tilde{F}_{\mu \nu} = B \cdot \tilde{E} \)
From $E_{\mu} = \alpha \mu \sigma x_{\alpha}$ we can get linearity transform for $E$ and $B$.

For a transformation from $k$ to $k'$, with $k'$ moving with $v$ along $x$, with respect to $k$,

\[
\begin{align*}
E_1' &= E_1 \\
E_2' &= \gamma (E_2 - \frac{v}{c} B_3) \\
E_3' &= \gamma (E_3 + \frac{v}{c} B_2) \\
B_1' &= B_1 \\
B_2' &= \gamma (B_2 + \frac{v}{c} E_3) \\
B_3' &= \gamma (B_3 - \frac{v}{c} E_2)
\end{align*}
\]

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**Kinematics**

\[\frac{d}{ds}\]

\[\gamma - \text{momentum} \quad p_\mu = m x_\mu = m u_\mu = (m \gamma \vec{u}, \text{emc} \gamma)\]

\[p_\mu^2 = m^2 u_\mu^2 = -m^2 c^2\]

\[\gamma - \text{force} \quad K_\mu = (\vec{K}, iK_0) \quad \text{"Minkowski force"}\]

**Newton's 2nd Law**

\[m \frac{d^2 x_\mu}{d\alpha^2} = K_\mu\]

\[\Rightarrow m \frac{d^2 u_\mu}{d\alpha^2} = \frac{dp_\mu}{ds} = K_\mu\]

\[p_\mu^2 = -m^2 c^2 \Rightarrow \frac{d}{ds} \left( p_\mu^2 \right) = p_\mu \frac{dp_\mu}{ds} = p_\mu K_\mu = 0\]

\[\Rightarrow m \gamma \vec{u} \cdot \vec{K} - m c^2 K_0 = 0 \quad \Rightarrow \quad K_0 = \frac{\vec{u} \cdot \vec{K}}{c^2} \]
Define the usual 3-force by
\[ \frac{dp}{dt} = F \]
\[ \frac{dp}{ds} = \mathbf{F} \text{ and } \frac{dp}{ds} = \mathbf{\delta F} = \mathbf{\delta F} \Rightarrow \mathbf{K} = \frac{\mathbf{\delta F}}{c} \]
\[ K_0 = \frac{\mathbf{\delta F}}{c} \]

Consider 4th component of Newton's 2nd law
\[ m \frac{d\mathbf{u}_y}{ds} = m \frac{d(\mathbf{ic} \cdot \mathbf{v})}{ds} = iK_0 = i \frac{\mathbf{\delta F}}{c} \]
\[ d(m\mathbf{r}) = \frac{\mathbf{\delta F}}{c^2} \cdot \mathbf{ds} = \frac{dt \cdot \mathbf{v} \cdot \mathbf{r} \cdot \mathbf{F}}{c^2} = \frac{dr \cdot \mathbf{F}}{c^2} \]

Work-energy theorem: \( d(m\gamma c^2) = d\mathbf{\delta F} \) \( \gamma \) work done
\[ \Rightarrow d(m\gamma c^2) \text{ is change in } \gamma \text{ kinetic energy} \]
\[ E = m\gamma c^2 \text{ is relativistic energy} \]

\[ \frac{\mathbf{p}}{c} = (p_x, p_y) \quad \mathbf{\dot{p}} = m\gamma \mathbf{\dot{v}} \quad E = m\gamma c^2 \]

\[ E = mc^2 \approx mc^2 \left(1 + \frac{1}{2} \frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2 \]

\[ \frac{df_{\mu}}{ds} = K_{\mu} \quad \text{therefore} \]

relativistic analog of Newton's 3rd law as well as law of conservation of energy
Lorentz force

\[ \frac{d \mu}{ds} = K \mu \]

What is the $K \mu$ that represents the Lorentz force and how can we write it in a covariant way?

$K \mu$ should depend on the fields $F_{\nu \mu}$ and the particle's trajectory $x_\mu$

As $\nu \to 0$, $K = \frac{q}{\nu}$

$K \mu$ can't depend directly on $x_\mu$ as it should be indep of origin of coords. So can depend only on $x_\mu$, $x_\nu$, etc.

As $\nu \to 0$, $K$ does not depend on the acceleration, so $K$ does not depend on $x_\mu$

$K \mu$ only depends on $F_{\nu \mu}$ and $x_\mu$

we need to form a 4-vector out of $F_{\nu \mu}$ and $x_\mu$ that is linear in the fields $F_{\nu \mu}$ and proportional to the charge $q$.

The only possibility is

\[ q = f (\nu^2) \ F_{\nu \mu} \ x_\nu \]
But \( \gamma^2 x^2 = c^2 \) is a constant. Choose \( f(x/y) = \frac{1}{c} \)

\[ K_j = \frac{q}{c} \mathbf{E}_j \times \mathbf{x}_j \text{ is only possibility} \]

This gives force

\[ \mathbf{F} = \frac{1}{c} \mathbf{k} \]

\[ F_i = \frac{1}{c} K_i = \frac{q}{yc} (F_j \dot{x}_j + F_m \dot{x}_m) \]

\[ = \frac{q}{yc} \left( \frac{\partial A_j}{\partial x_c} - \frac{\partial A_i}{\partial x_j} \right) \dot{x}_j + \frac{q}{yc} (-i \mathbf{E}_i)(ic\gamma) \]

\[ = \frac{q}{yc} \left[ E_{ijk} B_k \gamma \sigma_j \right] + \frac{q}{yc} \mathbf{E}_i \gamma \mathbf{c} \gamma \]

\[ = \frac{q}{c} \mathbf{E}_i + \frac{q}{c} \frac{\gamma}{c} B_k \]

\[ \mathbf{F} = \frac{q}{c} \mathbf{E} + \frac{q}{c} \frac{\gamma}{c} \times \mathbf{B} \]

Lorentz force is the same form in all inertial frames. No relativistic modification is needed.
Relativistic Larmor's formula

\[ P = \frac{\gamma}{2} \frac{8}{5} \left[ \langle a(t) \rangle \right]^2 \]

Consider inertial frame in which charge is instantaneously at rest. Call the rest frame K.

Power radiated in K is \( P = \frac{d \hat{E}(t)}{dt} \)

where \( \hat{E} \) is energy radiated. In K, the momentum density \( \frac{\partial \vec{P}}{\partial t} = \frac{1}{\sqrt{\gamma}} \frac{\partial \vec{E}}{\partial t} \times \vec{\beta} \sim \vec{E} \)

is in outward radial direction. Integrating over all directions, the radiated momentum vanishes \( \beta = 0 \)

Energy-momentum is a 4-vector \( (\hat{P}, \vec{i} \hat{E}) \)

To get radiated energy in original frame K we can use Lorentz transform

\[ \vec{E} = \gamma \left( \frac{\vec{E}}{c} - \frac{\vec{x}}{c} \frac{\partial \vec{E}}{\partial t} \right) \Rightarrow \vec{E} = \gamma \vec{E}^\prime \text{ as } \vec{P} = 0 \]

\[ ad \ dt = \gamma \delta \text{ time interval in K} \]

\( dS = 0 \) as charge stays at origin \( \vec{E} \)

\[ \frac{d\vec{E}}{dt} = \gamma \frac{\partial \vec{E}}{\partial t} = \gamma \frac{d\vec{E}}{d \tau} \Rightarrow \frac{\partial \vec{E}}{\partial S} = \gamma \frac{\partial \vec{E}}{\partial \tau} \]

Radiated power is Lorentz invariant!
in $\mathbb{R}$ we can use non-relativistic Larmor's formula

\[ P = \frac{2}{3} \frac{q \cdot a^2}{c^3} \]

$a$ is acceleration in $\mathbb{R}$

To write an expression without explicitly making mention of frame $\mathbb{R}$, we need to find a Lorentz invariant scalar that reduces to $a^2$ as $v \rightarrow 0$.

Only choice is $\alpha \mathbf{\mu}$, the 4-acceleration \[ \alpha \mathbf{\mu} = \frac{d\mathbf{\mu}}{ds} \]

\[ \alpha \mathbf{\mu} = \gamma \frac{d\mathbf{\mu}}{dt} = \gamma \mathbf{d} \left( \gamma \mathbf{v}, \mathbf{e}_3 \mathbf{v} \right) \]

\[ z = \gamma^2 \frac{d\mathbf{v}}{dt} + \mathbf{v} \frac{d\mathbf{v}}{dt} \]

\[ \alpha \mathbf{\mu} = \mathbf{e}_3 \gamma \frac{d\mathbf{v}}{dt} \]

\[ \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left( \frac{1}{\sqrt{1 - v^2/c^2}} \right) = \frac{\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}}{c^2 (1 - v^2/c^2)^{3/2}} = \frac{1}{c^2} \gamma^3 \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \]

as $\mathbf{v} \rightarrow 0$, $\gamma \rightarrow 1$, \[ \frac{d\mathbf{v}}{dt} \rightarrow 0 \]

so \[ z \rightarrow \frac{d\mathbf{v}}{dt} = a \]

\[ \alpha \mathbf{\mu}^2 \rightarrow |a|^2 \] as desired

\[ \alpha \mathbf{\mu}^2 \rightarrow |a|^2 \]

Relativistic Larmor's formula

\[ P = \frac{2}{3} \frac{q \cdot a^2}{c^3} \alpha \mathbf{\mu}^2 = \frac{2}{3} \frac{q \cdot a^2}{c^3} \left( \alpha \mathbf{\mu}^2 \right)^2 \]
\[ \alpha_\mu = \left( \gamma^2 \frac{d\nu}{dt} + \gamma \nu \frac{d\gamma}{dt} \right) \]

\[ \frac{d\nu}{dt} = \frac{1}{c^2} \gamma^3 \nu \cdot \vec{a} \]

\[ \alpha_\mu = \left( \gamma^2 \vec{a} + \gamma \nu \frac{1}{c^2} (\vec{v} \cdot \vec{a}) \frac{d\nu}{dt} \right) \cdot \vec{c} \gamma \nu \cdot \vec{a} \]

\[ \alpha_\mu^2 = \gamma^4 a^2 + \gamma^8 (\vec{v} \cdot \vec{a})^2 \frac{d\nu}{dt}^2 + 2 \gamma^6 (\vec{a} \cdot \vec{a})^2 - \gamma^8 (\vec{v} \cdot \vec{a})^2 \frac{c^2}{c^2} \]

\[ = \gamma^4 \left[ a^2 + \gamma^4 (\vec{v} \cdot \vec{a})^2 \left( \frac{d\nu}{dt}^2 - 1 \right) + 2 \gamma^2 (\vec{v} \cdot \vec{a})^2 \right] \]

\[ = \gamma^4 \left[ a^2 - \gamma^2 (\vec{v} \cdot \vec{a})^2 + 2 \gamma^2 (\vec{v} \cdot \vec{a})^2 \right] \]

\[ \alpha_\mu^2 = \gamma^4 \left[ a^2 + \gamma^2 (\vec{v} \cdot \vec{a})^2 \right] \]

As \( \vec{v} \to 0 \), \( \alpha_\mu^2 \to a^2 \)

\[ \alpha_\mu^2 = \vec{a}^2 \] 

Lorentz invariant

\( \vec{a} = \) acceleration in instantaneous rest

For a charge accelerating in linear motion, \( \mu = (\vec{v} \cdot \vec{a}) = \nu \vec{a} \)

\[ \alpha_\mu^2 = \gamma^4 a^2 \left( 1 + \frac{\gamma^2 c^2}{c^2} \right) = \gamma^6 a^2 \]

\[ p = \frac{\gamma^2 a^2}{3 c^3} \]

For a charge in circular motion \( (\vec{v} \cdot \vec{a}) = 0 \)

\[ \alpha_\mu^2 = \gamma^4 a^2 \]

\[ p = \frac{\gamma^4}{\frac{3}{c^3}} \]