From Coulomb to Maxwell

Electro dynamics is concerned with one particular attribute of matter - charge.

Experimentally, it was observed that certain bodies exert long range forces on each other that are certainly not gravitational - they are not proportional to the mass and they can be repulsive as well as attractive. The source of this new force was defined to be the "charge" of the object.

Electrostatics

Coulomb's Law - for charge \( q_1 \) at \( \vec{r}_1 \) and charge \( q_2 \) at \( \vec{r}_2 \), if separation \( \vec{r}_2 - \vec{r}_1 \) is much greater than the size of either charge, then

\[
F_{12} = \frac{k \cdot q_1 \cdot q_2}{r_{12}^2}
\]

\[
\hat{F}_{12} = \frac{\vec{r}_2 - \vec{r}_1}{r_{12}}
\]

force on 2 due to 1

central force - points from 1 to 2

inverse square law
$k_1$ is a universal constant of nature that determines the strength of the force when $q$ is expressed in terms of some arbitrary reference charge.

Since we only know about charge by measuring the Coulomb force, we are in principle free to choose $k_1$ to be anything we like — our choice then determines the units that charge is measured in.

In MKS system of units (same as SI system) charge is measured in the historical unit, the "coulomb." Then $k_1$ has the value

$$k_1 = \frac{1}{4\pi \varepsilon_0} = 10^{-7} \text{ C}^2,$$

where $c$ is speed of light in a vacuum. The units of $k_1$ are $\text{N} \cdot \text{m}^2/\text{Coul}^2$

In CGS system of units (also called esu - electrostatic units) one fixes $k_1 = 1$ and charge is measured in "statcoulombs." $k_1$ is taken dimensionless, so a statcoulomb is $(\text{Coul} \cdot \text{cm}^{-2})$.

Another reasonable modern choice would be to measure charge in integer multiples of the electron charge. This would yield a different value for $k_1$.

In this class we will be using CGS units. But we keep $k_1$ general for now.
**Superposition**

For charges $q_i$ at positions $\vec{r}_i$, the force on charge $Q$ at position $\vec{r}$ is:

$$\vec{F} = k \frac{Q}{4\pi \varepsilon_0} \sum_i q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}$$

*forces add linearly*

**Conservation of charge** - charge is neither created nor destroyed

$$\frac{d}{dt} \sum_i q_i = 0$$

where sum is over all charges in system

**Continuum charge density**

for charges $q_i$ at positions $\vec{r}_i$, define,

$$\rho(\vec{r}) = \sum_i q_i \delta(\vec{r} - \vec{r}_i)$$

$\delta(\vec{r} - \vec{r}_i)$ is Dirac $\delta$-function with properties:

$$\int_V d^3r \, \delta(\vec{r} - \vec{r}_i) = \begin{cases} 1 & \text{if } \vec{r}_i \in V \\ 0 & \text{otherwise} \end{cases}$$

$$\int_V d^3r \, f(\vec{r}) \delta(\vec{r} - \vec{r}_i) = \begin{cases} f(\vec{r}_i) & \text{if } \vec{r}_i \in V \\ 0 & \text{otherwise} \end{cases}$$

for any scalar function $f(\vec{r})$
\[ \sum_{r} g \cdot \int d^3r \cdot s(r, r_e) = \oint_{\partial V} \vec{d} \cdot \vec{E} \]

\[ = \text{total charge enclosed by volume } V \]

\[ \Rightarrow \rho \text{ has units of charge per volume} \]
\[ \Rightarrow s(r) \text{ has units of } 1/\text{vol} \]
\[ \frac{\vec{F}}{C} = k_1 \pi \int d^3r' \cdot \rho(r') \cdot \frac{\vec{r} - \vec{r}'}{|r - r'|^3} \]

We will often forget that \( \rho(r) \) is in principle made up of a distribution of point charges, and take it to be a smooth continuous function.

\[ \text{Charge conservation} \quad \frac{d}{dt} \int d^3r \cdot \rho(r) = 0 \]

assuming \( V \) is so big that it contains all the charge, and no charge flows through the surface of \( V \)
Electric Field

\[ \vec{E}(\vec{r}) \] is the force per unit charge that would be felt by an infinitesimal test charge \( q \) at position \( \vec{r} \).

\[ \vec{E}(\vec{r}) = \frac{1}{\cancel{\varepsilon_0}} \vec{F} = k_1 \int d^3\vec{r}' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad (\ast) \]

In principle, the above is a solution to all electrostatic problems. In practice, we may not always know \( \rho(\vec{r}) \), but may need to solve for it self-consistently with \( \vec{E} \). It will help to have another formulation of the above in terms of differential equations. We get these by using the divergence and curl of eq. (\ast) above to get (see proof later)

\[ \nabla \cdot \vec{E} = \frac{4\pi k_1}{\varepsilon_0} \rho \quad (1) \quad \text{Gauss' law} \]

\[ \nabla \times \vec{E} = 0 \quad (2) \left\langle \text{true only for statics!} \right\rangle \]

The proof of the above will follow on next page. We also can recast (1) and (2) in integral form as follows by Gauss' Theorem

\[ \int_V d^3\vec{r} \nabla \cdot \vec{E} = \oint_S d\vec{a} \cdot \vec{E} = \frac{4\pi k_1}{\varepsilon_0} \int_V d^3\rho \]

total charge enclosed in \( V \)

by Stokes

\[ \oint_S d\vec{a} \cdot \nabla \times \vec{E} = \int_C d\vec{l} \cdot \vec{E} = 0 \]
in above, \( n \) is outward pointing normal to surface \( S \)

\( \mathbf{d}^2 \) is differential tangent to curve \( C \) bounding surface \( S \)

\[ \text{Proof} \] that (1) and (2) follow from (\(*\))

First note that

\[
\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\mathbf{r}}{|\mathbf{r}|^2}.
\]

\[ -\nabla \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\nabla \frac{1}{r} \]

since \( \mathbf{r} = \mathbf{r} - \mathbf{r}' \), we have \( \nabla \) (which differentiates with respect to \( \mathbf{r} \))

is the same as \( \nabla \frac{1}{r} \) (which differentiates with respect to \( r \)).

So

\[ -\nabla \left( \frac{1}{r} \right) = -\frac{\hat{r}}{r^2} \left( \frac{1}{r} \right) \]

using \( \nabla \) in spherical coords.

So from Coulomb we have

\[
E(\mathbf{r}) = k_i \int d^3r' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}.
\]

\[ = -\nabla \left( k_i \int d^3r' \rho(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \]

\( \mathbf{E} \) is gradient of scalar function

\[
\nabla \times \nabla \phi = 0 \]

since the curl of a gradient always vanishes

\( \nabla \times \nabla \phi = 0 \) for any scalar function \( \phi \)
\[
\vec{\nabla} \cdot \vec{E} = -\nabla^2 \left( \vec{r}_1 \int d^3r' \frac{e(r')}{|r-r'|} \right) \quad \text{where } \nabla^2 = \vec{\nabla} \cdot \vec{\nabla}
\]

Consider
\[
\nabla^2 \left( \frac{1}{|r-r'|} \right)
\]
as before, define \( \vec{r} = \vec{r} - \vec{r}' \), so \( \vec{\nabla} = \vec{\nabla}_{r} \), and go to spherical coordinates centered at \( \vec{r} = 0 \).

\[
\nabla^2 \left( \frac{1}{|r-r'|} \right) = \nabla_r^2 \left( \frac{1}{r} \right)
\]

use expression for \( \nabla^2 \)
in spherical coordinates

\[
\frac{1}{r^2} \frac{d^2}{dr^2} \left( \frac{1}{r} \right)
\]

\[
= 0 \quad \text{for } r \neq 0
\]

\[
\text{singular at } r = 0
\]

So \( \nabla^2 \left( \frac{1}{r} \right) \) vanishes everywhere except at \( r = 0 \) to see what happens at \( r = 0 \), consider integrating over a sphere \( V \) of radius \( R \) centered at the origin

\[
\int_V d^3r \ \nabla^2 \left( \frac{1}{r} \right) = \int_V d^3r \ \vec{\nabla} \cdot \vec{\nabla} \left( \frac{1}{r} \right) = \int_S da \ \vec{n} \cdot \vec{\nabla} \left( \frac{1}{r} \right)
\]

using Gauss' Theorem

\[
\int_S da \ \vec{n} \cdot \vec{\nabla} \left( \frac{1}{r} \right) = \frac{1}{r^2} \left( \frac{1}{r} \right) = -\frac{1}{r^2} \quad \text{constant on surface } S
\]

so

\[
\int_S da \ \vec{n} \cdot \vec{\nabla} \left( \frac{1}{r} \right) = 4\pi R^2 \left( -\frac{1}{R^2} \right) = -4\pi
\]
Above was integrating over a sphere, but we would get same result if integrated over any volume containing \( \mathbf{r} = 0 \).

Let \( S \) be a sphere of radius \( R \) and \( S' \) be any surface.

Then by Gauss theorem,

\[
\int_{S'} d^3r \cdot \mathbf{v}(\mathbf{r}) = \int_S dA \mathbf{n} \cdot \mathbf{v}(\mathbf{r}) = -\int_S dA \mathbf{n} \cdot \mathbf{v}(\mathbf{r})
\]

\[
= 0 \quad \text{since} \quad \nabla^2(\mathbf{v}) = 0 \quad \text{everywhere in } V'
\]

\[
\Rightarrow \int_S dA \mathbf{n} \cdot \mathbf{v}(\mathbf{r}) = \int_S dA \mathbf{n} \cdot \mathbf{v}(\mathbf{r})
\]

\[
\Rightarrow \int_{V'} d^3r \nabla^2(\mathbf{v}) = \int_{V} d^3r \nabla^2(\mathbf{v})
\]

So we conclude:

\[
\int_{V} d^3r \nabla^2(\mathbf{v}) = \begin{cases} -4\pi & \text{if } \mathbf{r} = 0 \text{ in } V \\ 0 & \text{if } \mathbf{r} = 0 \text{ not in } V \end{cases}
\]

\[
\Rightarrow \nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta(\mathbf{r}) \quad \text{Dirac delta function}
\]

\[
\nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta(\mathbf{r} - \mathbf{r}')
\]
So now
\[ \nabla \cdot \vec{E} = -k_i \int d^3r \, p(r') \sqrt{\frac{1}{|r-r'|}} \]

\[ = -k_i \int d^3r \, p(r') (-4\pi) \delta(r-r') \]

\[ = 4\pi k_i \rho(r) \text{ by property of } \delta \text{-function} \]

proof is done!

we have shown that
\[ \vec{E}(\vec{r}) = k_i \int d^3r' \, p(\vec{r}') \frac{\vec{r}-\vec{r}'}{|r-r'|} \Rightarrow \begin{cases} \nabla \cdot \vec{E} = 4\pi k_i \rho \\ \nabla \times \vec{E} = 0 \end{cases} \]

is the reverse true? i.e. is the formulation in terms of partial differential equations completely equivalent to Coulomb's law? Yes! because of \textit{Helmholtz's Theorem}.

\textit{Helmholtz Theorem} of vector calculus — if one specifies the divergence and curl of a vector function, and boundary conditions (here \( E \to 0 \) as \( r \to \infty \) and one is away from all charges), then vector function is uniquely determined.
\textbf{Helmholtz Theorem}

Suppose \[ \begin{align*}
\nabla \cdot \vec{E} &= f(\vec{r}) \\
\nabla \times \vec{E} &= g(\vec{r}) \\
\end{align*} \]
for \( \vec{r} \in V \)
and \( f \) or \( g \) known functions.

\[ \vec{E}(\vec{r}) = \vec{E}_0(\vec{r}) \] for \( \vec{r} \in \partial V \) the boundary of \( V \).

Suppose we had two different solutions \( \vec{E} \) and \( \vec{E}' \) to the above equations.

Then consider \( \vec{G} = \vec{E} - \vec{E}' \).

\( \vec{G} \) must satisfy
\[ \begin{align*}
\nabla \cdot \vec{G} &= 0 \\
\nabla \times \vec{G} &= 0 \\
\vec{G} &= 0 \\
\end{align*} \] for \( \vec{r} \in \partial V \).

Now \( \nabla \times \vec{G} = 0 \Rightarrow \vec{G} = \nabla \phi \) for some scalar potential \( \phi \).

\[ \begin{align*}
\nabla \cdot \vec{G} &= 0 \\
\nabla^2 \phi &= 0 \quad \text{in } V \\
\phi &= 0 \quad \text{on } \partial V \\
\end{align*} \]

by properties of harmonic function.

for harmonic function \( \phi(\vec{r}) = \frac{1}{4\pi} \int_{\partial S} d\vec{a} \phi(\vec{r}') \) where \( S \) is sphere of radius \( R \) centered at \( \vec{r} \).

\( \phi(\vec{r}) \) can have no extremum inside a volume \( V \Rightarrow \) all extrema must be boundary \( \partial V \). If \( \partial V \) is empty, \( \vec{d} \phi = 0 \) \( \Rightarrow \nabla = 0 \)