From Coulomb to Maxwell

Electrodynamics is concerned with one particular attribute of matter - charge

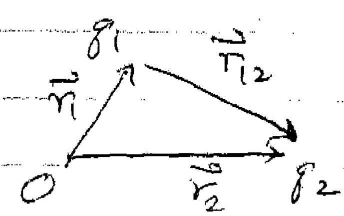
Experimentally it was observed that certain bodies exert long range forces on each other that are certainly not gravitational - they are not proportional to the mass and they can be repulsive as well as attractive. The source of this new force was defined to be the "charge" of the object

Electrostatics

Coulomb's Law - for charge q_1 at $\vec{r}_1$ and charge q_2 at $\vec{r}_2$, if separation $|\vec{r}_2 - \vec{r}_1|$ is much greater than the size of either charge, then

$$\vec{F}_{12} = k_1 q_1 q_2 \frac{\hat{r}_{12}}{r_{12}^2} \qquad \vec{r}_{12} = \vec{r}_2 - \vec{r}_1 \qquad \hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$$

force on 2 due to 1



central force - points from 1 to 2
inverse square law

k_1 is a <u>universal</u> constant of nature that determines the strength of the force when q is expressed in terms of some arbitrary reference charge.

Since we only know about charge by measuring the Coulomb force, we are in principle free to choose k_1 to be anything we like – our choice then <u>determines</u> the units that charge is measured in.

In MKS system of units (same as SI system) charge is measured in the historical unit, the "coulomb." Then k_1 has the value $k_1 = \frac{1}{4\pi\epsilon_0} = 10^{-7}c^2$, where c is speed of light in a vacuum. The units of k_1 are $\text{Nt}\cdot\text{m}^2/\text{coul}^2$

In CGS system of units (also called esu – electrostatic units) one fixes $k_1 = 1$ and charge is measured in "statcoulombs." k_1 is taken dimensionless, so statcoulomb $= (\text{dyne-cm}^2)^{1/2}$

Another reasonable modern choice would be to measure charge in integer multiples of the electron charge. This would yield a different value for k_1

In this class we will be using CGS units. But we keep k_1 general for now.

Superposition

For charges q_i at positions $\vec{r}_i$, the force on charge Q at position $\vec{r}$ is

$$\vec{F} = k_1 Q \sum_i q_i \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3}$$ forces add linearly

Conservation of charge

– charge is neither created nor destroyed

$$\frac{d}{dt}\sum_i q_i = 0$$ where sum is over all charges in system

Continuum charge density

for charges q_i at positions $\vec{r}_i$, define,

$$\rho(\vec{r}) = \sum_i q_i \, \delta(\vec{r}-\vec{r}_i)$$

$\delta(\vec{r}-\vec{r}_i)$ is Dirac δ-function with properties:

$$\int_V d^3r \; \delta(\vec{r}-\vec{r}_i) = \begin{cases} 1 & \text{if } \vec{r}_i \in V \\ 0 & \text{otherwise} \end{cases}$$

$$\int_V d^3r \; f(\vec{r}) \, \delta(\vec{r}-\vec{r}_i) = \begin{cases} f(\vec{r}_i) & \text{if } \vec{r}_i \in V \\ 0 & \text{otherwise} \end{cases}$$

for any scalar function $f(\vec{r})$

$$\int_V d^3r\, \rho(\vec{r}) = \sum_i q_i \int d^3r\, \delta(\vec{r}-\vec{r}_i)$$

$$= Q_{encl} \quad \text{total charge enclosed by volume } V$$

$\Rightarrow$ ρ has units of charge per volume
$\Rightarrow$ $\delta(\vec{r})$ has units of $1/\text{vol}$

<u>Coulomb</u>

$$\vec{F} = k_1 Q \int d^3r'\, \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

We will often forget that $\rho(\vec{r})$ is in principle made up of a distribution of point charges, and take it to be a smooth continuous function.

<u>charge conservation</u> – $\dfrac{d}{dt} \displaystyle\int_V d^3r\, \rho(\vec{r}) = 0$

assuming V is so big that it contains all the charge, and no charge flows through the surface of V

Electric Field

$\vec{E}(\vec{r})$ is the force per unit charge that would be felt by an infinitesimal test charge δq at position $\vec{r}$.

$$\vec{E}(\vec{r}) = \frac{1}{\delta q}\vec{F} = k_1 \int d^3r' \, \rho(\vec{r}\,') \, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \qquad (*)$$

In principle, the above is solution to all electrostatic problems. In practise, we may not always know $\rho(\vec{r})$, but may need to solve for it self consistently with $\vec{E}$. It will help to have another formulation of the above in terms of differential equations. We get these by taking the divergence and curl of eq $(*)$ above to get (see proof later)

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= 4\pi k_1 \rho \quad (1) \quad \text{Gauss' Law} \\ \vec{\nabla}\times\vec{E} &= 0 \quad (2) \leftarrow \text{true only for } \underline{\text{statics}}\,! \end{aligned}}$$

The proof of the above will follow on next page.

We also can recast (1) and (2) in integral form as follows

by Gauss' Theorem

$$\int_V d^3r \, \vec{\nabla}\cdot\vec{E} = \oint_S da \, \hat{n}\cdot\vec{E} = 4\pi k_1 \int_V d^3r \, \rho$$

total charge enclosed in V

So

$$\oint_S da \, \hat{n}\cdot\vec{E} = 4\pi k_1 Q_{encl}$$

by Stokes Theorem

$$\int_S da \, \hat{n}\cdot\vec{\nabla}\times\vec{E} = \oint_C d\vec{l}\cdot\vec{E} = 0$$

in above, $\hat{n}$ is outward pointing normal to surface S
$d\vec{l}$ is differential tangent to curve C bounding surface S

Proof that (1) and (2) follow from (*)

First Note that $\boxed{\dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}\left(\dfrac{1}{|\vec{r}-\vec{r}'|}\right)}$

to see this, let $\vec{\mathscr{r}} \equiv \vec{r}-\vec{r}'$, and do the calculation in spherical coords centered at $\vec{\mathscr{r}}=0$.

$$\frac{r-r'}{|\vec{r}-\vec{r}'|^3} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$-\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\vec{\nabla}\frac{1}{\mathscr{r}}$$

since $\vec{\mathscr{r}} = \vec{r}-\vec{r}'$, we have $\vec{\nabla}$ (which differentiates with respect to $\vec{r}$) is the same as $\vec{\nabla}_{\mathscr{r}}$ (which differentiates with respect to $\vec{\mathscr{r}}$)

So $$-\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\hat{\mathscr{r}}\frac{d}{d\mathscr{r}}\left(\frac{1}{\mathscr{r}}\right)$$ using $\vec{\nabla}$ in spherical coords

$$= \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

So from Coulomb we have

$$\vec{E}(\vec{r}) = k_1\int d^3r'\,\rho(\vec{r}')\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

$$= -\vec{\nabla}\left(k_1\int d^3r'\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)$$ $\vec{E}$ is gradient of scalar function

$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$ since the curl of a gradient always vanishes
$\vec{\nabla}\times\vec{\nabla}\phi = 0$ for any scalar function f

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\left(k_1\int d^3r'\,\frac{\rho(r')}{|\vec{r}-\vec{r}'|}\right) \qquad \text{where } \nabla^2 = \vec{\nabla}\cdot\vec{\nabla}$$

Consider

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

as before, define $\vec{\imath} = \vec{r}-\vec{r}'$, so $\vec{\nabla}_r = \vec{\nabla}_{\imath}$, and go to spherical coords centered at $\vec{\imath} = 0$.

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \nabla_{\imath}^2\left(\frac{1}{\imath}\right)$$

use expression for ∇^2 in spherical coords

$$= \frac{1}{\imath}\frac{d^2}{d\imath^2}\,\imath\left(\frac{1}{\imath}\right)$$

$$= \begin{cases} 0\,! & \text{for } \imath \neq 0 \\ \text{singular} & \text{at } \imath = 0 \end{cases}$$

So

$\nabla^2\left(\frac{1}{\imath}\right)$ vanishes everywhere except at $\imath = 0$

to see what happens at $\imath = 0$, consider integrating over a sphere V of radius R centered at the origin

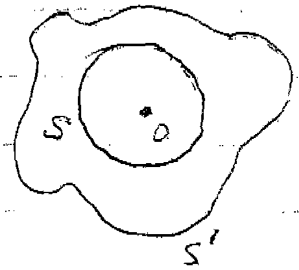
$$\int_V d^3r\,\nabla^2\left(\frac{1}{\imath}\right) = \int_V d^3r\,\vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{\imath}\right) = \oint_S da\,\hat{n}\cdot\vec{\nabla}\left(\frac{1}{\imath}\right)$$

using Gauss' Theorem

integrand $\hat{n}\cdot\vec{\nabla}\left(\frac{1}{\imath}\right) = \frac{d}{d\imath}\left(\frac{1}{\imath}\right) = -\frac{1}{\imath^2}$ is constant on surface S so

$$\oint_S da\,\hat{n}\cdot\vec{\nabla}\left(\frac{1}{\imath}\right) = 4\pi R^2\left(\frac{-1}{R^2}\right) = -4\pi$$

Above was integrating over a sphere, but we would get same result if integrated over any volume containing $\vec{r}=0$.



S is sphere of radius R
S' is any surface

let $\bar{V}'$ be volume between S and S'
Then by Gauss Theorem

$$\int_{V'} d^3r\, \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \oint_{S'} da\, \hat{n}\cdot\vec{\nabla}\left(\frac{1}{r}\right) - \oint_{S} da\, \hat{n}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$$

$= 0$ since $\nabla^2\left(\frac{1}{r}\right) = 0$ everywhere in V'

$$\Rightarrow \oint_{S'} da\, \underset{\shortparallel}{\hat{n}}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \oint_{S} da\, \underset{\shortparallel}{\hat{n}}\cdot\vec{\nabla}\left(\frac{1}{r}\right)$$

$$\Rightarrow \int_{V'} d^3r\, \nabla^2\left(\frac{1}{r}\right) = \int_{V} d^3r\, \nabla^2\left(\frac{1}{r}\right)$$

V' bounded by S' — V bounded by S

So we conclude: for any volume V

$$\int_V d^3r\, \nabla^2\left(\frac{1}{r}\right) = \begin{cases} -4\pi & \text{if } \vec{r}=0 \text{ in } V \\ 0 & \text{if } \vec{r}=0 \text{ not in } V \end{cases}$$

$$\Rightarrow \nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r})$$ Dirac delta function

$$\boxed{\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')}$$

acts only on $\vec{r}$

So now

$$\vec{\nabla}\cdot\vec{E} = -k_1\int d^3r'\,\rho(r')\,\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$= -k_1\int d^3r'\,\rho(\vec{r}')\,(-4\pi)\,\delta(\vec{r}-\vec{r}')$$

$$= 4\pi k_1\,\rho(\vec{r}) \quad \text{by property of } \delta\text{-function}$$

proof is done!

we have shown that

$$\vec{E}(\vec{r}) = k_1\int d^3r'\,\rho(\vec{r}')\,\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi k_1\rho \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$$

is the reverse true? ie is the formulation in terms of partial differential equations completely equivalent to Coulomb's Law? Yes! because of Helmholtz's Theorem.

Helmholtz Theorem of vector calculus — if one specifies the divergence and curl of a vector function, and boundary conditions (here $E \to 0$ as $r \to \infty$ and one is away from all charges), then vector function is uniquely determined

Helmholtz Theorem

Suppose
$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{E} &= f(\vec{r})\\ \vec{\nabla}\times\vec{E} &= \vec{g}(\vec{r})\end{aligned}\right\}\ \text{for } \vec{r}\in V$$
f and g known functions

$$\vec{E}(\vec{r}) = \vec{h}(\vec{r}) \quad \text{for } \vec{r}\in S \text{ the boundary of } V$$

Suppose we had two different solutions $\vec{E}$ and $\vec{E}'$ to the above equations.

Then consider $\vec{G} \equiv \vec{E} - \vec{E}'$.

$\vec{G}$ must satisfy

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{G} &= 0\\ \vec{\nabla}\times\vec{G} &= 0\end{aligned}\right\}\ \text{for } \vec{r}\in V$$

$$\vec{G} = 0 \quad \text{for } \vec{r}\in S$$

Now $\vec{\nabla}\times\vec{G} = 0 \Rightarrow \vec{G} = \vec{\nabla}\phi$ for some scalar potential ϕ.

$$\vec{\nabla}\cdot\vec{G} = 0 \Rightarrow \left.\begin{aligned}\nabla^2\phi &= 0 \quad \text{in } V\\ \phi &= 0 \quad \text{on } S\end{aligned}\right\} \Rightarrow \phi = 0$$

by properties of harmonic function

for harmonic function $\phi(\vec{r}) = \frac{1}{4\pi R^2}\oint_S da'\,\phi(\vec{r}\,')$ where S is sphere of radius R centered at $\vec{r}$

$\Rightarrow \phi(\vec{r})$ can have no extremum inside a volume $V \Rightarrow$ all extrema must lie on boundary of V. If $\phi = 0$ on boundary $\Rightarrow \phi_{max} = \phi_{min} = \phi = 0$