Transverse + Longitudinal Parts of vector functions

To prove the preceding claim, $\vec{j} = \vec{j}_{\parallel} + \vec{j}_{\perp}$, where $\vec{\nabla}\times\vec{j}_{\parallel} = 0$ and $\vec{\nabla}\cdot\vec{j}_{\perp} = 0$, we first digress to prove Helmholtz Theorem.

Helmholtz Theorem: For a vector function $\vec{f}(\vec{r})$ if one knows the divergence and curl of $\vec{f}$ then one can ~~uniquely~~ uniquely determine $\vec{f}$ itself.

That is, if

$$\vec{\nabla}\cdot\vec{f} = 4\pi D(\vec{r})$$ where $D(\vec{r})$ is a known scalar function

$$\vec{\nabla}\times\vec{f} = 4\pi\vec{C}(\vec{r})$$ where $\vec{C}(\vec{r})$ is a known vector function

~~Then one can solve for~~

And if well defined boundary conditions on $\vec{f}$ are known (here we will assume $\vec{f}(\vec{r})\to 0$ as $\vec{r}\to\infty$)

Then there is a unique solution for $\vec{f}(\vec{r})$.

We prove this by construction!

Assume a solution of the form

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$ where φ is a scalar and $\vec{W}$ a vector

Now we show that we can find such a solution

First consider

$$\vec{\nabla}\cdot\vec{f} = -\nabla^2\varphi + \vec{\nabla}\cdot(\vec{\nabla}\times\vec{W}) = -\nabla^2\varphi + 0 = 4\pi D(\vec{r})$$

So $-\nabla^2\varphi = 4\pi D(\vec{r})$ This is just Poisson's eqn we saw in electrostatics

Solution when $\varphi(\vec{r})\to 0$ as $r\to\infty$ is given by

$$\boxed{\varphi(\vec{r}) = \int d^3r' \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

Coulomb-like integral solution

Now consider

$$\vec{\nabla}\times\vec{f} = -\vec{\nabla}\times\vec{\nabla}\varphi + \vec{\nabla}\times(\vec{\nabla}\times\vec{W}) = 0 - \nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot\vec{W})$$
$$= 4\pi\vec{C}(\vec{r})$$

Choose a gauge in which $\vec{\nabla}\cdot\vec{W}=0$ (just like Coulomb gauge in magnetostatics)

Then $-\nabla^2\vec{W} = 4\pi\vec{C}(\vec{r})$

$$\boxed{\vec{W}(\vec{r}) = \int d^3r' \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

just like solution for vector pot $\vec{A}$ in magnetostatics

So we have constructed a solution

$$f(\vec{r}) = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$

$$= -\vec{\nabla}\int d^3r' \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|} + \vec{\nabla}\times\int d^3r' \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

where $\vec{\nabla}\cdot\vec{f} = 4\pi D$ and $\vec{\nabla}\times\vec{f} = 4\pi\vec{C}$

Note: For above solution to be well defined, the integrals must converge. They will converge if the "sources" $D(\vec{r})$ and $\vec{C}(\vec{r})$ are sufficently "localized" in space, ie $D(\vec{r}) \to 0$, $\vec{C}(\vec{r}) \to 0$ sufficently fast as $\vec{r} \to \infty$.

Now we show that the above solution is <u>unique</u>.

Suppose there was another solution $\vec{g}$ such that

$$\vec{\nabla}\cdot\vec{g} = 4\pi D \quad \text{and} \quad \vec{\nabla}\times\vec{g} = 4\pi\vec{C}$$

Consider $\vec{h} \equiv \vec{f} - \vec{g}$ then

$$\vec{\nabla}\cdot\vec{h} = 0 \quad \text{and} \quad \vec{\nabla}\times\vec{h} = 0$$

Can show that only such $\vec{h}$ that also has $\vec{h}(\vec{r}) \to 0$ as $\vec{r} \to \infty$ is $\vec{h} \equiv 0$, so $\vec{g} = \vec{f}$ and solution is unique.

As a consequence of Helmholtz theoran we have also shown the following

① Any vector function $\vec{f}$ can be written in terms of a scalar and vector potential

$$\vec{f} = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{W}$$

or equivalently

(2) Any vector function $\vec{f}$ can be written in terms of a curl free and a divergenceless part

$$\vec{f} = \vec{f}_\parallel + \vec{f}_\perp \qquad \text{where} \qquad \begin{array}{ll} \vec{\nabla}\times\vec{f}_\parallel = 0 & \text{curlfree} \\ \vec{\nabla}\cdot\vec{f}_\perp = 0 & \text{divergenceless} \end{array}$$

$$\text{where} \begin{cases} \vec{f}_\parallel(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}) = -\vec{\nabla}\displaystyle\int \frac{d^3r'}{4\pi}\,\frac{[\vec{\nabla}'\cdot\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \\ \vec{f}_\perp(\vec{r}) = \vec{\nabla}\times\vec{W}(\vec{r}) = \vec{\nabla}\times\displaystyle\int \frac{d^3r'}{4\pi}\,\frac{[\vec{\nabla}'\times\vec{f}(\vec{r}')]}{|\vec{r}-\vec{r}'|} \end{cases}$$

where in above we used $\vec{D}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\cdot\vec{f}(\vec{r}')$

$$\vec{C}(\vec{r}') = \frac{1}{4\pi}\vec{\nabla}'\times\vec{f}(\vec{r}')$$

~~where~~ $\vec{f}_\parallel$ is called the <u>longitudinal</u> part of $\vec{f}$

$\vec{f}_\perp$ is called the <u>transverse</u> part of $\vec{f}$

To understand the reason for these names, we need to consider the <u>Fourier transforms</u>

Above can be generalized to situations where $\vec{f}$ satisfies other boundary conditions, say has a specified value on a given boundary surface. One first replaces $\frac{1}{|\vec{r}-\vec{r}'|}$ by the appropriate <u>Greens function</u> — see more to come!

Discussion regarding Fourier transforms

$$\vec{f}(\vec{r}) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \vec{f}(\vec{k}) \quad \text{Fourier transf}$$

$$\vec{f}(\vec{k}) = \int_{-\infty}^{\infty} d^3r \, e^{-i\vec{k}\cdot\vec{r}} \vec{f}(\vec{r}) \quad \text{inverse transf}$$

Some special cases well worth remembering

① Transform of Dirac function

$$\delta_{\vec{r}_0}(\vec{k}) \equiv \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \delta(\vec{r}-\vec{r}_0) = e^{-i\vec{k}\cdot\vec{r}_0}$$

$$\Rightarrow \quad \delta(\vec{r}-\vec{r}_0) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \delta_{\vec{r}_0}(\vec{k})$$

$$\boxed{\delta(\vec{r}-\vec{r}_0) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}}$$

Or letting $\vec{r} \leftrightarrow \vec{k}$ in the above

$$\delta(\vec{k}-\vec{k}_0) = \int \frac{d^3r}{(2\pi)^3} e^{i\vec{r}\cdot(\vec{k}-\vec{k}_0)}$$

② Transform of Coulomb potential $\frac{1}{|\vec{r}-\vec{r}'|}$

We know

$$\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

Suppose $f(\vec{k}) \equiv \int d^3r \, e^{-i\vec{k}\cdot\vec{r}} \frac{1}{|\vec{r}-\vec{r}'|}$ is the

Fourier transf of $\frac{1}{|\vec{r}-\vec{r}'|}$

Substitute
$$\begin{cases} \dfrac{1}{|\vec{r}-\vec{r}'|} = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, f(\vec{k}) \\ \delta(\vec{r}-\vec{r}') = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \end{cases}$$

into above Poisson equation

$$\underbrace{\nabla^2 \int \frac{d^3k}{(2\pi)^3}}\, e^{i\vec{k}\cdot\vec{r}}\, f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}$$

operates only on $\vec{r}$
so move inside integral

$$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = \vec{\nabla}\cdot\left(\vec{\nabla} e^{i\vec{k}\cdot\vec{r}}\right)$$

① $$\vec{\nabla} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i \frac{\partial}{\partial x_i} e^{i\vec{k}\cdot\vec{r}} = \sum_{i=1}^{3} \hat{x}_i\, i k_i\, e^{i\vec{k}\cdot\vec{r}}$$

$$= i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}$$ where $\hat{x}_1, \hat{x}_2, \hat{x}_3 = \hat{x}, \hat{y}, \hat{z}$

② $$\vec{\nabla}\cdot\left(i\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\right) = (i\vec{k})\cdot(i\vec{k})\, e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

so $$\nabla^2 e^{i\vec{k}\cdot\vec{r}} = -k^2 e^{i\vec{k}\cdot\vec{r}}$$

Poisson eqn then gives

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} (-k^2)\, f(\vec{k}) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}\cdot\vec{r}'}$$

$$\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-k^2 f(\vec{k})\right] = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}} \left[-4\pi\, e^{-i\vec{k}\cdot\vec{r}'}\right]$$

As is true for Fourier series, so it is true for Fourier transforms: If two functions are equal, then their Fourier transforms are equal.

$$\Rightarrow \quad -k^2 f(\vec{k}) = -4\pi e^{-i\vec{k}\cdot\vec{r}\,'}$$

$$\boxed{f(\vec{k}) = \frac{4\pi}{k^2} e^{-i\vec{k}\cdot\vec{r}\,'}}$$

$\Rightarrow$ is the Fourier transform of $\dfrac{1}{|\vec{r}-\vec{r}\,'|}$

Electrostatic

$-\nabla^2\Phi = 4\pi\rho$ with $\vec{E} = -\vec{\nabla}\phi$ (statics only)

physical meaning of the potential ϕ

work done to move a test charge δq from $\vec{r}_1$ to $\vec{r}_2$ in presence of an electric field $\vec{E}$ is

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{F}$$

where $\vec{F}$ is the force required to move the charge. Since $\vec{E}$ exerts a force $\delta q\,\vec{E}$ on the charge, $\vec{F}$ must counter-balance this electric force so we can move the charge quasi statically $\Rightarrow \vec{F} = -\delta q\,\vec{E}$

$$W_{12} = -\delta q \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{E} = \delta q \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l} \cdot \vec{\nabla}\phi = \delta q\left[\phi(\vec{r}_2) - \phi(\vec{r}_1)\right]$$

$$\phi(\vec{r}_2) - \phi(\vec{r}_1) = \frac{W_{12}}{\delta q}$$

difference in potential between two points is the work per unit charge to move a test charge between the two points

Green's Functions – part I

$$-\nabla^2 \phi = 4\pi \rho$$

We already know that for a point charge q at position $\vec{r}'$, ie $\rho(\vec{r}) = q\,\delta(\vec{r}-\vec{r}')$, the solution to the above is

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|} \qquad \text{ie} \quad -\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = 4\pi\,\delta(\vec{r}-\vec{r}')$$

We call the special solution for a point source the <u>Green function</u> for the differential operator

$$-\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$$

$G(\vec{r},\vec{r}')$ gives the potential at position $\vec{r}$ due to a unit source at position $\vec{r}'$

Generally, one also has to specify a desired boundary condition for the Green function on the boundary of the system.

For the Coulomb solution for a point charge the implicit boundary condition is that the potential vanish infinitely far from the charge

$$G(\vec{r},\vec{r}') \to 0 \quad \text{as} \quad |\vec{r}-\vec{r}'| \to \infty$$

boundary of the system is taken to infinity

If one knows the Green's function, then one can find the solution for any distribution of sources $\rho(\vec{r})$

$$\phi(\vec{r}) = \int d^3r' \, G(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,')$$

proof: $$-\nabla^2\phi = \int d^3r' \left[-\nabla^2 G(\vec{r},\vec{r}\,')\right] \rho(\vec{r}\,')$$

$$= \int d^3r' \left[4\pi\,\delta(\vec{r}-\vec{r}\,')\right] \rho(\vec{r}\,')$$

$$= 4\pi\,\rho(\vec{r})$$

We will return to concept of Greens function when we discuss solution of Poisson's eqn in a finite volume

We will also see Greens functions again when we discuss solution of the inhomogeneous wave equation.

The Coulomb problem as a boundary value problem

Consider a conducting sphere of radius R with net charge q (as $R \to 0$ we get a point charge).
What is $\phi(\vec{r})$? What is $E(\vec{r})$?

Review: Properties of conductors in electrostatics

1) $\vec{E}=0$ inside conductor - if $\vec{E}\neq 0$ then a current $\vec{j}=\sigma\vec{E}$ flows and it is not statics (σ is conductivity)
2) $\rho=0$ inside conductor - if $\vec{E}=0$ inside, then $\vec{\nabla}\cdot\vec{E}=4\pi\rho=0$
3) Any net charge on the conductor must lie on the surface - follows from (2)
4) $\phi=$ constant throughout conductor - if $\vec{E}=0$ then $\vec{E}=-\vec{\nabla}\phi \Rightarrow \phi$ is constant
5) Just outside the conductor, $\vec{E}$ is $\perp$ to surface.
   - If $\vec{E}$ has a component $\parallel$ to surface then it exerts a force on electrons at the surface leading to a surface current - so would not be static

For conducting sphere, $\rho=0$ for $r>R$ and $r<R$
all charge is on the surface $\Rightarrow \nabla^2\phi=0$ for $\begin{cases} r>R \\ r<R \end{cases}$

spherical symmetry $\Rightarrow$ expect spherically symmetric solution

$\Rightarrow \phi(\vec{r})$ depends only on $r=|\vec{r}|$

⇒ Solve Laplace's eqn by writing ∇^2 in spherical coords. Only the radial terms do not vanish.

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 0$$

$$r^2 \frac{d\phi}{dr} = -C_0 \quad \text{a constant}$$

$$\frac{d\phi}{dr} = -\frac{C_0}{r^2}$$

$$\phi(r) = \frac{C_0}{r} + C_1 \quad , \quad C_1 \text{ a constant}$$

"outside" $r > R$ $\quad \phi^{out}(r) = \frac{C_0^{out}}{r} + C_1^{out}$

"inside" $r < R$ $\quad \phi^{in}(r) = \frac{C_0^{in}}{r} + C_1^{in}$

solution "outside" does not necessarily go smoothly into the solution "inside" because of the charge layer at $r = R$ that separates the two regions. We need to determine the constants C_0^{in}, C_0^{out}, C_1^{in}, C_1^{out} by applying <u>boundary conditions</u> corresponding to the physical situation.

① For $r > R$, assume $\phi \to 0$ as $r \to \infty$ – boundary condition at infinity

$$\Rightarrow C_1^{out} = 0$$

$\phi^{out}(r) = \frac{C_0^{out}}{r}$ recover the expected Coulomb form.

2) For $r < R$.

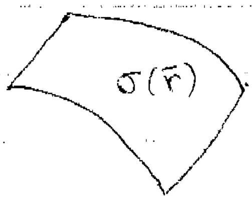
i) We could use the fact that the region $r < R$ is a conductor with ϕ = constant to conclude $C_0^{in} = 0$

ii) or, if we were dealing with a charged shell instead of a conductor, we could argue as follows: no charge at origin $r = 0$ $\Rightarrow$ expect ϕ should be finite at origin $\Rightarrow C_0^{in} = 0$

So $\phi^{in}(r) = C^{in}$ a constant

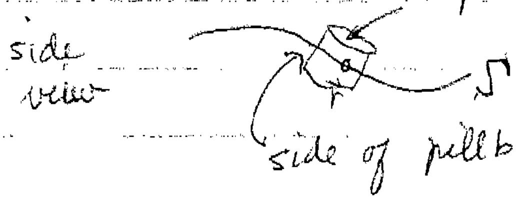
3) Now we need boundary condition at $r = R$ where "inside" and "outside" meet.

Review: Electric field and potential at a surface charge layer



← a general surface S with surface charge density $\sigma(\vec{r})$ for $\vec{r}$ on S. $\sigma(\vec{r})da$ is total charge in area da on surface

i) Take "Gaussian pillbox" surface about point $\vec{r}$ on the surface S'



Gauss' Law in integral form $\oint_S da\, \hat{n} \cdot \vec{E} = 4\pi Q_{enclosed}$

expect $\vec{E}$ is finite $\Rightarrow$ contribution from sides of pillbox vanish as $dl \to 0$.

$$\oint_S da\, \hat{n}\cdot\vec{E} = \int_{top} da\, \hat{n}\cdot\vec{E} + \int_{bottom} da\, \hat{n}\cdot\vec{E}$$

$$= \left(\hat{n}^{top}\cdot\vec{E}^{top} + \hat{n}^{bottom}\cdot\vec{E}^{bottom}\right) da \qquad \text{since } da \text{ is small}$$

$\vec{E}^{top}$ is electric field at $\vec{r}$ just above the surface S

$\vec{E}^{bottom}$ is electric field at $\vec{r}$ just below the surface S

$\hat{n}^{top} \equiv \hat{n}$ is outward normal on top.

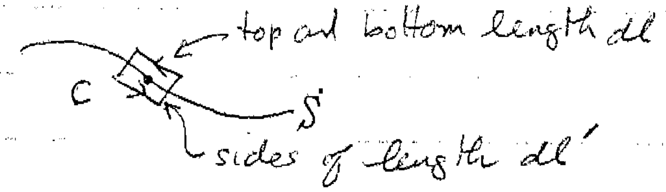
$\hat{n}^{bottom} = -\hat{n}$ is outward normal on bottom

$$\Rightarrow \left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot\hat{n}\, da = 4\pi Q_{enclosed} = 4\pi\sigma(\vec{r})\, da$$

$$\boxed{\left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot\hat{n} = 4\pi\sigma(\vec{r})}$$ <u>discontinuity</u> in <u>normal</u> component of $\vec{E}$

ii) Take "Amperian loop" C at surface about point $\vec{r}$

side view



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \oint_C d\vec{l}\cdot\vec{E}$ since $\vec{E}$ is finite at surface, if take sides $dl' \to 0$ their contribution to integral vanishes

$$\Rightarrow \oint_C d\vec{l}\cdot\vec{E} = \boxed{\left(\vec{E}^{top} - \vec{E}^{bottom}\right)\cdot d\vec{l} = 0}$$

where $d\vec{l}$ is any infinitesimal tangent to the surface at $\vec{r}$.