iii) \( \vec{E} = -\nabla \phi \Rightarrow \phi (r_2) - \phi (r_1) = -\int_{r_1}^{r_2} \vec{E} \cdot d\vec{r} \)

Take \( r_2 \) just above \( r \) on surface \( \int_{r} \vec{E} \cdot d\vec{r} \rightarrow 0 \)

Take \( r_1 \) just below \( r \) on surface \( \int_{r} \vec{E} \cdot d\vec{r} \rightarrow 0 \)

Since \( \vec{E} \) is finite \( \Rightarrow \int \vec{E} \cdot d\vec{r} \rightarrow 0 \)

\( \Rightarrow \phi_{\text{top}} = \phi_{\text{bottom}} \)

potential \( \phi \) is continuous at surface charge layer

Can rewrite (i) as

\[
\left(-\nabla \phi_{\text{top}} + \nabla \phi_{\text{bottom}}\right) \cdot \hat{n} = 4\pi \sigma
\]

\[
\frac{-\partial \phi_{\text{top}}}{\partial m} + \frac{\partial \phi_{\text{bottom}}}{\partial m} = 4\pi \sigma
\]

\( \hat{n} \) directional derivative of \( \phi \) in direction \( \hat{n} \)

Discontinuity in normal derivative of \( \phi \) at surface

Applying to conducting sphere

\( \phi \) continuous \( \Rightarrow \phi^{\text{in}}(R) = \phi^{\text{out}}(R) \)

\( C_1^{\text{in}} = \frac{C_{\text{out}}}{R} \)

Only one maximum or \( \phi \)
normal derivative of $\phi$ is discontinuous

$$- \frac{\partial \phi_{\text{top}}}{\partial m} + \frac{\partial \phi_{\text{bottom}}}{\partial m} = 4\pi \sigma$$

where $\vec{n} = \hat{r}$, the radial direction

$$\left[ - \frac{d \phi_{\text{out}}}{dr} + \frac{d \phi_{\text{in}}}{dr} \right]_{r=R} = 4\pi \sigma$$

but \( \frac{d \phi_{\text{in}}}{dr} = 0 \) as \( \phi_{\text{in}} = \text{constant} \)

$$\left. - \frac{d \phi_{\text{out}}}{dr} \right|_{r=R} = 4\pi \sigma$$

change $\sigma$ is uniformly distributed on surface at $R$

$$- \frac{1}{dr} \left( \frac{C_{\text{out}}}{r} \right)_{r=R} = \frac{C_{\text{out}}}{R} = 4\pi \sigma = 4\pi \left( \frac{\sigma R^2}{4\pi R^2} \right) = \frac{\sigma}{R^2}$$

$$\Rightarrow C_{\text{out}} = \frac{\sigma}{R} \quad C_{\text{in}} = \frac{C_{\text{out}}}{r} = \frac{\sigma}{r}$$

$$\phi(r) = \begin{cases} \frac{\sigma}{R} & r < R \quad \text{inside} \\ \frac{\sigma}{r} & r > R \quad \text{outside} \end{cases}$$

$$\Rightarrow \vec{E} = -\vec{\nabla} \phi = -\frac{d \phi}{dr} = \begin{cases} 0 & r < R \quad \text{inside} \\ \frac{\sigma}{r^2} & r > R \quad \text{outside} \end{cases}$$

we get familiar Coulomb solution!
Summary
we can use the preceding solution for \( \phi \) out as solving Laplace's equation \( \nabla^2 \phi = 0 \) subject to a specified boundary condition on the normal derivative of \( \phi \) at the boundary \( r = R \) of the "outside" region of the system.

Alternate problem:
Another physical situation would be to connect a conducting sphere to a battery that charge the sphere to a fixed voltage \( \phi_0 \) (stat. volts?) with respect to ground \( \phi = 0 \) at \( r \to \infty \).

As before, outside the sphere \( \phi = \frac{C_0}{r} \)
Now the boundary condition is to specify the value of \( \phi \) on the boundary of the outside region, i.e.
\[ \phi(R) = \phi_0 \]
\[ \Rightarrow \frac{C_0}{R} = \phi_0 \quad , \quad C_0 = \phi_0 R \]
\[ \phi(r) = \phi_0 \frac{R}{r} \]
(from preceding solution, we knew that charging the sphere to voltage \( \phi_0 \) (stat. volts) induces a net charge \( q = \phi_0 R \) on it.)
These two versions of the conducting sphere problem are examples of a more general *boundary value problem*.

Solve \( \nabla^2 \phi = 0 \) in a given region of space subject to one of the following two types of boundary conditions on the bounding surfaces of the region:

i) Neumann boundary condition

\[ \frac{\partial \phi}{\partial n} \text{ - normal derivative of } \phi \text{ is specified on the boundary surface} \]

ii) Dirichlet boundary condition

\[ \phi \text{ - value of } \phi \text{ is specified on the boundary surfaces} \]

If the boundary surfaces consist of disjoint pieces, it is possible to specify either (i) or (ii) on each piece separately to get a *mixed boundary value problem*. 
Some more problems

An infinite conducting wire of radius \( R \) with linear charge density \( \lambda \) = charge per unit length

Surface charge \( \sigma = \frac{\lambda}{2\pi R} \)

Expect cylindrical symmetry \( \Rightarrow \phi \) depends only on cylindrical coord \( r \).

\[ \nabla^2 \phi = 0 \quad \text{for} \quad r > R, \quad r < R \]

Use \( \nabla^2 \) in cylindrical coords — only radial term non-vanishing

\[ \nabla^2 \phi = \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) = 0 \]

\[ r \frac{d\phi}{dr} = C_0 \quad \text{constant} \]

\[ \frac{d\phi}{dr} = \frac{C_0}{r} \]

\[ \phi(r) = C_0 \ln r + C_1 \quad \text{const} \]

Note: one cannot now choose \( \phi \to 0 \) as \( r \to \infty \)!

One needs to fix zero of \( \phi \) at some other radius. A convenient choice is \( r = R \), but any other choice could also be made.
\[ \phi^{\text{out}} = \frac{C_0}{R} \ln r + C_1 \]

\[ \phi^{\text{in}} = \frac{C_0}{R} \ln r + C_1 \]

\[ \phi^{\text{in}} = \text{const} \text{ m conductor } \Rightarrow C_0 = 0 \]

or \( \phi^{\text{in}} \) should not diverge as \( r \to 0 \) \( \Rightarrow C_0 = 0 \)

So \( \phi^{\text{in}} = C_1 \) constant

Boundary condition at \( r = R \)

\[ \left[ \frac{d\phi^{\text{out}}}{dr} + \frac{d\phi^{\text{in}}}{dr} \right]_{r=R} = 4\pi \sigma \]

\[ -\frac{C_0}{R} = 4\pi \sigma = 4\pi \left( \frac{\lambda}{2\pi R} \right) = \frac{2\lambda}{R} \]

\[ C_0 = -2\lambda \]

\[ \phi^{\text{out}}(r) = -2\lambda \ln r + C_1^{\text{out}} \]

Continuity of \( \phi \)

\[ \phi^{\text{in}}(R) = \phi^{\text{out}}(R) \Rightarrow C_1 = -2\lambda \ln R + C_1^{\text{out}} \]

Remaining const \( C_1^{\text{out}} \) is not too important as it is just a common additive constant to both \( \phi^{\text{in}} \) and \( \phi^{\text{out}} \) does not change \( E = -\nabla \phi \).

If use the condition \( \phi(R) = 0 \) then we can solve for \( C_1^{\text{out}} \).
\[ 0 = -2\lambda \ln R + c_1^{\text{out}} \quad \Rightarrow \quad c_1^{\text{out}} = 2\lambda \ln R \]

\[ \Rightarrow \phi (r) = \begin{cases} -2\lambda \ln \left( \frac{r}{R} \right) & r > R \\ 0 & r < R \end{cases} \]

\[ E (r) = \begin{cases} \frac{2\lambda}{r} & r > R \\ 0 & r < R \end{cases} \]

**infinite conducting half space**

\[ \sigma \text{ uniform surface charge density} \]

\[ \text{expect } \phi \text{ depends only on } x \]

\[ \nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \]

\[ \Rightarrow \begin{cases} \phi^{\geq} (x) = \phi^{\leq} (x) + c_1^{\geq} & x > 0 \\ \phi^{\leq} (x) = \phi^{\geq} (x) + c_1^{\leq} & x < 0 \end{cases} \]

for \( x < 0 \), \( \phi = \text{const at conductor} \Rightarrow c_1^{\leq} = 0 \)

at \( x = 0 \), \( \phi \) continuous \( \Rightarrow \phi^{\geq} (0) = \phi^{\leq} (0) \)

\[ c_1^{\leq} = c_1^{\geq} \]

\[ \frac{d\phi}{dx} \text{ discontinuous} \Rightarrow \]

\[ \left. -\frac{d\phi}{dx} \right|_{x=0} = 4\pi \sigma \]

\[ \Rightarrow c_1^{\geq} = -4\pi \sigma \]

\[ \Rightarrow \phi (x) = \begin{cases} -4\pi \sigma x + c_1^{\geq} & x > 0 \\ c_1^{\geq} & x < 0 \end{cases} \]

\[ \text{const } c_1^{\geq} \text{ do not change value of } E \]
as for the wire, we cannot choose $\phi \to 0$ as $x \to \infty$. We can set $\phi \to \pm \infty$.

\[
-\nabla^2 \phi = \hat{E} = \begin{cases} 4\pi S x & x > 0 \\ 0 & x < 0 \end{cases}
\]

\underline{infinite charged plane}

Similar to previous problem, but now no conductor at $x < 0$, just free space on both sides of the charged plane at $x = 0$.

\begin{align*}
\nabla^2 \phi &= \frac{d^2 \phi}{dx^2} = 0 \\
\phi^+ &= C_0^+ x + C_i^+ \quad x > 0 \\
\phi^- &= C_0^- x + C_i^- \quad x < 0
\end{align*}

\underline{continuity of $\phi$ at $x = 0$}

\[
\phi^+(0) = \phi^-(0) \Rightarrow C_i^+ = C_i^-
\]

\underline{discontinuity of $d\phi/dx$ at $x = 0$}

\[
-\frac{d\phi^+}{dx} + \frac{d\phi^-}{dx} = 4\pi S
\]

\[
- C_0^+ + C_0^- = 4\pi S
\]

Define $\bar{C}_0 = \frac{C_0^+ + C_0^-}{2}$

Then we can write
\[
\begin{align*}
\phi &= \begin{cases} 
-2\pi \sigma x + \overline{C}_0 x + C_i^+ & x > 0 \\
2\pi \sigma x + \overline{C}_0 x + C_i^- & x < 0
\end{cases}
\end{align*}
\]

\[
\frac{d\phi}{dx} = \overline{E} = \begin{cases} 
(2\pi \sigma - \overline{C}_0) \hat{x} & x > 0 \\
(-2\pi \sigma - \overline{C}_0) \hat{x} & x < 0
\end{cases}
\]

Const. \(C_i\) does not affect \(\overline{E}\) - additive const to \(\phi\)

\(\overline{C}_0\) represents const uniform electric field \(-\overline{C}_0 \hat{x}\) that exists independently of the charged surface.

If we assumed that all \(\overline{E}\) fields are just those arising from the plane, then we can set \(\overline{C}_0 = 0\).

Equivalently, if the plane is the only source of \(\overline{E}\), then we expect \(\phi\) depends only on \(|x|\) by symmetry.

\[\Rightarrow \quad C_i^+ = -C_i^- \quad \text{and again } \overline{C}_0 = 0. \]

In this case

\[
\phi(x) = \begin{cases} 
-2\pi \sigma x & x > 0 \\
2\pi \sigma x & x < 0
\end{cases}
\]

we also set \(C_i = 0\) here correspondingly \(\phi(0) = 0\)

\[
\overline{E}(x) = \begin{cases} 
2\pi \sigma \hat{x} & x > 0 \\
-2\pi \sigma \hat{x} & x < 0
\end{cases}
\]

\(\overline{E}\) is constant but oppositely directed on either side of the charged plane.
Green's Theorem, Uniqueness, Green function - part II

We want to show that the boundary value problem we described is well posed - i.e. there is a unique solution. We start by deriving Green's theorem.

Consider \( \int d^3r \, \nabla \cdot \mathbf{A} = \oint_{\partial S} \mathbf{A} \cdot \hat{n} \) - Gauss theorem.

Let \( \mathbf{A} = \mathbf{\phi} \mathbf{v} \phi \), \( \phi, \psi \) any two scalar functions.

\[ \Rightarrow \, \nabla \cdot \mathbf{A} = \mathbf{\phi} \nabla^2 \phi + \nabla \phi \cdot \mathbf{v} \phi \]

\[ \mathbf{\phi} \nabla \phi \cdot \hat{n} = \mathbf{\phi} \frac{\partial \phi}{\partial n} \]

\[ \Rightarrow \int d^3r \, \left( \mathbf{\phi} \nabla^2 \phi + \mathbf{\phi} \nabla \phi \cdot \mathbf{v} \phi \right) = \oint_{\partial S} \mathbf{\phi} \frac{\partial \phi}{\partial n} \] - Green's 1st identity.

Let \( \phi \leftrightarrow \psi \)

\[ \int d^3r \, \left( \mathbf{\psi} \nabla^2 \psi + \mathbf{\psi} \nabla \psi \cdot \mathbf{v} \psi \right) = \oint_{\partial S} \mathbf{\psi} \frac{\partial \psi}{\partial n} \]

Subtracting the two equations gives:

\[ \int d^3r \, \left( \mathbf{\phi} \nabla^2 \phi - \mathbf{\psi} \nabla^2 \psi \right) = \oint_{\partial S} \left( \mathbf{\phi} \frac{\partial \phi}{\partial n} - \mathbf{\psi} \frac{\partial \psi}{\partial n} \right) \] - Green's 2nd identity.

Apply Green's 2nd identity with \( \mathbf{\phi} = \frac{1}{r-r'} \), \( r' \) is the integration variable, \( \mathbf{\psi} \) the scalar potential with \( \nabla^2 \phi = -4\pi \rho \). Use \( \nabla^2 \psi = \nabla^2 \phi = -4\pi \delta(r-r') \)

\[ \int d^3r' \left[ \mathbf{\phi}(r') \left[ -4\pi \delta(r-r') \right] - \left( \frac{1}{r-r'} \right) \left( -4\pi \delta(r-r') \right) \right] \]

\[ = \oint_{\partial S} \left[ \mathbf{\phi} \frac{\partial \phi}{\partial n} \left( \frac{1}{r-r'} \right) - \frac{1}{r-r'} \frac{\partial \phi}{\partial n} \right] \]
If \( \vec{r} \) lies within the volume \( V \), then

\[
\phi(\vec{r}) = \frac{\int d^3r' \, \rho(\vec{r}')}{V} + \oint_{\partial V} \frac{d\vec{a}'}{4\pi} \left[ \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \phi}{\partial m'} - \frac{2}{|\vec{r} - \vec{r}'|} \frac{\partial^2 \phi}{\partial m' \partial m''} \right]
\]

Note: if \( \vec{r} \) lies outside the volume \( V \), then

\[
0 = \frac{\int d^3r' \, \rho(\vec{r}')}{V} + \oint_{\partial V} \frac{d\vec{a}'}{4\pi} \left[ \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \phi}{\partial m'} - \frac{2}{|\vec{r} - \vec{r}'|} \frac{\partial^2 \phi}{\partial m' \partial m''} \right]
\]

potential from a
surface charge density
potential from a
surface diffuse layer of
diffuse strength density

From (\( \star \)), if \( S \to \infty \) and \( E \to \frac{\partial \phi}{\partial m'} \to 0 \) faster than \( \frac{1}{r} \),
then the surface integral vanishes and we recover
Coulomb's law \( \phi(\vec{r}) = \sqrt{\int d^3r' \, \rho(\vec{r}')/|\vec{r} - \vec{r}'|} \)

(\( \star \)) gives the generalization of Coulomb's law to a system
with a finite boundary.

For a charge free volume \( V \), i.e. \( \rho(\vec{r}) = 0 \) in \( V \),
the potential everywhere is determined by the
potential and its normal derivative on the surface.

But one cannot in general freely specify both
\( \phi \) and \( \frac{\partial \phi}{\partial m'} \) on the boundary surface since the
resulting \( \phi \) from (\( \star \)) would not in general obey
Laplace's equation \( \nabla^2 \phi = 0 \).
Specifying both $\phi$ and $\frac{\partial \phi}{\partial n}$ on surface is known as
"Cauchy" boundary conditions — for Laplace's eqn,
Cauchy b.c. overspecify the problem + a solution
cannot in general be found.

Uniqueness

If we have a system of charges in vol $V$,
and either the potential $\phi$, or its normal
derivative $\frac{\partial \phi}{\partial n}$, is specified on the surfaces of $V$,
then there is a unique solution to Poisson's equation
inside $V$. Specifying $\phi$ is known as Dirichlet
boundary conditions. Specifying $\frac{\partial \phi}{\partial n}$ is known as
Neumann boundary conditions.

Proof. Suppose we had two solutions $\phi_1$, and $\phi_2$,
both with $-\nabla^2 \phi = 4\pi \rho$ inside $V$, and obeying
specified b.c. on surface of $V$.

Define $U = \phi_2 - \phi_1 \Rightarrow \nabla^2 U = 0$ inside $V$

and $U = 0$ on surface $S$ — for Dirichlet b.c.
or $\frac{\partial U}{\partial n} = 0$ on surface $S$ — for Neumann b.c.

Use Green's 1st identity with $\phi = \psi = U$

$$\int_V \left( \nabla^2 U + \nabla U \cdot \nabla U \right) \, d^3r = \oint_S U \frac{\partial U}{\partial n}$$

as $\nabla^2 U = 0$ as $U \circ \frac{\partial U}{\partial n} = 0$
\[ \int d^3r \left| \nabla u \right|^2 = 0 \quad \Rightarrow \quad \nabla u = 0 \quad \Rightarrow \quad u = \text{const} \]

For Dirichlet b.c., \( u = 0 \) on surface \( S \), so \( \text{const} = 0 \) and \( \phi_1 = \phi_2 \). Solution is unique.

For Neumann b.c., \( \phi_1 \) and \( \phi_2 \) differ only by an arbitrary constant. Since \( E = -\nabla \phi \), the electric fields \( E_1 = -\nabla \phi_1 \) and \( E_2 = -\nabla \phi_2 \) are the same.

**Biharmonic**: If boundary surface \( S \) consists of several disjoint pieces, then solution is unique if specify \( \phi \) on some pieces and \( \frac{\partial \phi}{\partial n} \) on other pieces.

Solution of Poisson's equation with both \( \phi \) and \( \frac{\partial \phi}{\partial n} \) specified on the same surface \( S \) (Cauchy b.c.) does not in general exist, since specifying either \( \phi \) or \( \frac{\partial \phi}{\partial n} \) alone is enough to give a unique solution.