Spherical Coordinates

\[ \nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

\[ \phi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \]

\[ r^2 \nabla^2 \phi = \frac{\Theta \Phi}{\sin \theta} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R \Phi}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d \Theta}{d\theta} \right) + \frac{R \Theta}{\sin \theta} \frac{1}{\phi} \frac{d^2 \Phi}{d\phi^2} = 0 \]

\[ \frac{r^2}{\sin \theta} \Phi \nabla^2 \phi = \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d \Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \]

- depends only on \( r \) and \( \theta \)
- \( = -\text{const} \)
- depends only on \( \phi \)
- \( = \text{const} \)

Take \[ \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -m^2 \]

\[ \Rightarrow \Phi = e^{\pm i m \phi} \]

- \( m \) integer for \( 2\pi \) periodicity
- \( \phi \)

\[ \Rightarrow \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d \Theta}{d\theta} \right) = m^2 \]

\[ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Omega} \frac{d}{d\theta} \left( \sin \theta \frac{d \Omega}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0 \]

- depends only on \( r \)
- \( = \text{const} \)
- \( \Theta \)
- \( = -\text{const} \)
Call the const \( l(l+1) \)

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - l(l+1) = 0
\]

Solutions are of the form \( R(r) = a_r r^l + b_r r^{-(l+1)} \)

Substitute \( R \) to verify

\[
\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{d}{dr} \left( r^2 \left( l a_r r^{l-1} - (l+1) b_r r^{-l-2} \right) \right)
\]

\[
= \frac{d}{dr} \left( l a_r r^{l+1} - (l+1) b_r r^{-l} \right)
\]

\[
= l(l+1) a_r r^l + (l+1) b_r r^{-(l+1)} = l(l+1) R
\]

For \( \theta \):

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = -l(l+1)
\]

Let \( x = \cos \theta \)

\[
dx = -\sin \theta \, d\theta
\]

\[
\frac{dx}{\sin \theta} = -d\theta
\]

Solutions for \(-1 \leq x \leq 1\) correspond to \( l \geq 0 \) integers

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d\theta}{dx} \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] \theta = 0
\]

Called generalized Legendre Equation - Solutions are called the associated Legendre functions.

Ordinary Legendre polynomials are solutions for \( m = 0 \)
For the special case \( m = 0 \), the solution has azimuthal symmetry and \( \Phi \) does not depend on the angle \( \theta \), (i.e., rotational symmetry about \( \hat{z} \) axis),

We want the solutions to

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d \Phi}{dx} \right] + \ell (\ell + 1) \Phi = 0
\]

The solutions are known as the Legendre polynomials, \( P_\ell(x) \).

They are given, for \( \ell \) integer, by

\[
P_\ell(x) = \frac{1}{2^\ell \ell!} \left( \frac{d}{dx} \right)^\ell (x^2 - 1)^\ell
\]

Rodriguez's formula.

The lowest \( \ell \) polynomials are

\[
\begin{align*}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2} (3x^2 - 1) \\
P_3(x) &= \frac{1}{2} (5x^3 - 3x)
\end{align*}
\]

In general, \( P_\ell(x) \) is a polynomial of order \( \ell \) with only even powers of \( x \) even and only odd powers of \( x \) odd. \( \Rightarrow P_\ell(x) \begin{cases} \text{even } \ell \text{ for } \ell \text{ even} \\ \text{odd } \ell \text{ for } \ell \text{ odd} \end{cases} \)

\( P_0(x) \) is normalized so that \( P_0(1) = 1 \).
Note: Legendre polynomials are only for integer \( l \geq 0 \). What about solutions for non integer \( l \)?

The \( P_l(x) \) give one solution for each integer \( l \). But \( P_l(x) \) are defined by a \( 2^{\text{nd}} \) order differential equation – shouldn’t there be a \( 2^{\text{nd}} \) independent solution for each \( l \)?

It turns out that these “\( 2^{\text{nd}} \)” solutions, as well as solutions for non integer \( l \), all blow up at either \( x = -1 \) or \( x = 1 \), i.e. at \( \theta = 0 \) or \( \theta = \pi \). They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2.

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval \(-1 \leq x \leq 1\).

\[
\int_{-1}^{1} P_l(x) P_m(x) \, dx = \int_0^\pi \sin \theta \, P_l(\cos \theta) \, P_m(\cos \theta) \, d\theta = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases}
\]

\( \Rightarrow \) we can expand any function \( f(\theta) \), \( 0 \leq \theta \leq \pi \), as a linear combination of the \( P_l(\cos \theta) \).

This is the reason they are useful for solving problems of Laplace’s eqn with spherical boundary surfaces.
For \( m \neq 0 \), the solutions to
\[
\frac{d^2}{dx^2} \left[ (1-x^2) \frac{d}{dx} \right] + \left[ \ell (1+x) - \frac{m^2}{1-x^2} \right] \Theta = 0
\]
are the associated Legendre functions \( P^m \ell (x) \).

For \( P^m \ell (x) \) to be finite on interval \([-1, 1]\), one again finds that \( \ell \) must be integer \( \ell \geq 0 \), and integer \( m \) must satisfy \( |m| \leq \ell \), i.e., \( m = -\ell, -\ell+1, \ldots, 0, \ldots, \ell-1, \ell \).

For each \( \ell \) and \( m \), there is only one such non-divergent solution.

It is typical to combine the solutions \( P^m \ell (\cos \theta) \) to the \( \theta \)-part of the equation with the \( \Phi^m \ell (\phi) = e^{im \phi} \) solutions to the \( \phi \)-part of the equation to define the spherical harmonics
\[
Y^m \ell (\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} P^m \ell (\cos \theta) e^{im \phi}
\]
The \( Y^m \ell \) are orthogonal,
\[
\int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi \, Y^m \ell^* (\theta, \phi) Y^m \ell (\theta, \phi) = \delta_{\ell \ell'} \delta_{mm'}
\]
and are a complete set of basis functions for expanding any function \( f(\theta, \phi) \) defined on the surface of a sphere.
Examples with azimuthal symmetry $m = 0$

General solution to $\nabla^2 \phi = 0$ can be written in form

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos \theta)$$

determine the $A_l$ and $B_l$ from the boundary conditions of the particular problem.

1. Suppose one is given $\phi(R, \theta) = \phi_0(\theta)$ on surface of sphere of radius $R$.

To find solution of $\nabla^2 \phi = 0$ inside sphere

$\phi$ should not diverge at origin $\Rightarrow B_l = 0$

for all $l$

$$\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$\Rightarrow \quad \phi(R, \theta) = \phi_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta)$$

$$\Rightarrow \quad \int_0^{\pi} d\theta \sin \theta \quad \phi_0(\theta) P_m(\cos \theta) = \sum_{l=0}^{\infty} A_l R^l \int_0^{\pi} d\theta \sin \theta P_l(\cos \theta) P_m(\cos \theta)$$

$$= \sum_{l=0}^{\infty} A_l R^l \frac{2}{2l+1} \delta_{lm}$$

$$\Rightarrow A_m = \frac{2m+1}{2R^m} \int_0^{\pi} d\theta \sin \theta \quad \phi_0(\theta) P_m(\cos \theta)$$

saves solution
To find solution of $\nabla^2 \phi = 0$ outside sphere

if require $\phi \to 0$ as $r \to 0$, then $A_e = 0$ for all $l$

$$
\phi(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B\ell}{r^{\ell+1}} P_{\ell}(\cos \theta)
$$

$$
\phi(R, \theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} \frac{B\ell}{R^{\ell+1}} P_{\ell}(\cos \theta)
$$

Gives solution

$$
B_m = \frac{2m+1}{2} R^{m+1} \int_0^{\pi} \sin^2 \phi_0(\theta) P_m(\cos \theta) d\theta
$$

(2)

$$
B_m = A m R^{2m+1}
$$

Suppose one is given surface charge density

$\sigma(\theta)$ fixed on surface of sphere of radius $R$.

What is $\phi$ inside and outside?

From previous example

$$
\phi(r, \theta) = \begin{cases} 
\sum_{\ell=0}^{\infty} A_e r^\ell P_{\ell}(\cos \theta) & r < R \\
\sum_{\ell=0}^{\infty} \frac{B\ell}{r^{\ell+1}} P_{\ell}(\cos \theta) & r > R 
\end{cases}
$$

boundary conditions at $r = R$ on surface

(i) $\phi$ continuous

$$
\sum_{\ell=0}^{\infty} \left[ A_e R^\ell - \frac{B\ell}{R^{\ell+1}} \right] P_{\ell}(\cos \theta) = 0
$$
If an expansion in Legendre polynomials vanishes for all \( \theta \), then each coefficient in the expansion must vanish:

\[
A_e R^e = \frac{B_e}{R^{e+1}} \Rightarrow B_e = A_e R^{2e+1}
\]

(iii) jump in electric field at \( \sigma \):

\[
- \frac{\partial \Phi^a^{\text{out}}}{\partial r} \bigg|_{r=R} + \frac{\partial \Phi^a^{\text{in}}}{\partial r} \bigg|_{r=R} = 4\pi \sigma
\]

\[
\Rightarrow \sum_{\ell=0}^{\infty} \left[ \frac{(\ell+1) B_e}{R^{\ell+2}} + \ell A_e R^{\ell-1} \right] p_{\ell}^{\text{in}}(\cos \theta) = 4\pi \sigma
\]

\[
\Rightarrow \sum_{\ell=0}^{\infty} \left[ \frac{(\ell+1) A_e R^{2\ell+1}}{R^{\ell+2}} + \ell A_e R^{\ell-1} \right] p_{\ell}^{\text{in}}(\cos \theta)
\]

\[
\Rightarrow \sum_{\ell=0}^{\infty} (2\ell+1) R^{\ell-1} A_e p_{\ell}^{\text{in}}(\cos \theta) = 4\pi \sigma
\]

\[(2m+1) R^{m-1} A_m \left( \frac{2}{2m+1} \right) = 4\pi \int_0^\pi d\phi \sin \theta \sigma(\theta) P_m(\cos \theta)
\]

\[
A_m = \frac{4\pi}{2m-1} \int_0^\pi \sin^2 \theta \sigma(\theta) P_m(\cos \theta)
\]
Suppose \( \sigma(\theta) = k \cos \theta \). What is \( \phi \)?

Note \( \sigma(\theta) = k P_1(\cos \theta) \)

Hence only \( A_1 \neq 0 \) by orthogonality of \( P_1(\cos \theta) \).

\[
A_1 = \frac{4\pi k}{2} \int_0^\pi \sin \theta P_1(\cos \theta) P_1(\cos \theta) \, d\theta
\]

\[
= \frac{4\pi k}{2} \left( \frac{2}{2+1} \right) = \frac{4\pi k}{3}
\]

\[
\Rightarrow \phi(r, \theta) = \begin{cases} 
\frac{4\pi k}{3} r \cos \theta & r < R \\
\frac{4\pi k}{3} \frac{R^3}{r^2} \cos \theta & r > R 
\end{cases}
\]

We will see that potential outside the sphere is that of an ideal dipole with dipole moment

\[
p = \frac{4\pi k}{3} R^3
\]

Inside the sphere, the potential \( \phi = \frac{4\pi k}{3} \frac{y}{r} \)

where \( y = r \cos \theta \). The electric field inside the sphere is therefore the constant

\[
E = -\nabla \phi = -\frac{4\pi k}{3} \frac{\partial}{\partial r}
\]
outside the sphere the field is

\[ \vec{E} = -\vec{\nabla} \phi = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} \]

\[ = \frac{8 \pi}{3} k R^3 \cos \theta \hat{r} + \frac{4 \pi}{3} k R^3 \sin \theta \hat{\theta} \]

\[ \vec{E} = \frac{4 \pi R^3 k}{3} \frac{1}{r^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] \]

dipole field
Physical example with $\sigma(\theta) = 2 \cos \theta$.

Two spheres of radius $R$, with equal but opposite uniform charge densities $\sigma$ and $-\sigma$, displaced by small distance $d \ll R$.

Surface charge $\sigma$ builds up due to displacement. This is a uniformly "polarized" sphere.

Surface charge is $\sigma(\theta) = \sigma S \sin \theta = \sigma d \cos \theta$.

\[ \sigma(\theta) = \sigma d \cos \theta \]

Total dipole moment is $$(\sigma d) \frac{4}{3} \pi R^3$$

Polarization = \[\frac{\text{dipole moment}}{\text{volume}}\] = $\sigma d$.

\[ \vec{E} \text{ field inside a uniformly polarized sphere is constant.} \]

\[ \vec{E} = -\sigma d \frac{4 \pi}{3} \]
Grounded conducting sphere in uniform electric field $\vec{E} = E_0 \hat{z}$

as $r \to \infty$ far from sphere, $\vec{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$

Boundary conditions:

\[ \phi(r, \theta) = 0 \]
\[ \phi(r \to \infty, \theta) = -E_0 \cos \theta \]

Solution outside sphere has the form

\[ \phi(r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l (\cos \theta) \]

From boundary condition as $r \to \infty$ we have

$A_l = 0 \quad \text{all } l \neq 1$

$A_1 = -E_0 \quad \text{since } P_1 (\cos \theta) = \cos \theta$

\[ \phi(r, \theta) = -E_0 \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l (\cos \theta) \]

From $\phi(r, \theta) = 0$ we have

\[ 0 = -E_0 R \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l (\cos \theta) \]

\[ \Rightarrow B_l = 0 \quad \text{all } l \neq 1 \]

\[ \frac{B_1}{R^2} = E_0 R \Rightarrow B_1 = +E_0 R^3 \]
\[ \phi(r, \theta) = -E_0 \left( r - \frac{R^2}{r^2} \right) \cos \theta \]

1st ten is just potential \(-E_0 \cos \theta\) of the uniform applied electric field.

2nd ten is potential due to the induced surface charge on the surface—it is a dipole field.

Induced charge density is

\[ 4\pi \sigma(\theta) = -\frac{\partial \phi}{\partial r} \bigg|_{r=R} = E_0 \left( 1 + \frac{2R^3}{r^3} \right) \cos \theta \]

\[ = 3E_0 \cos \theta \]

\[ \sigma(\theta) = \frac{3}{4\pi} E_0 \cos \theta \]

like uniformly polarized sphere

\[ \kappa = \frac{3E_0}{4\pi} \]

From (1) we know that the field inside the sphere due to this \( \sigma \) is just

\[ -\frac{1}{2} \kappa \hat{r} = -\frac{1}{3} \pi \frac{3E_0 \hat{\theta}}{4\pi} \]

\[ = -E_0 \hat{\theta} \] This is just what is required so that the total field in the conducting sphere vanishes.

Can check that outside the sphere, \( \hat{\vec{E}} = -\hat{\nabla} \phi \) is normal to surface of sphere at \( r = R \).
Behavior of fields near conical hole or sharp tip.

We now want to solve the $\nabla^2 \phi = 0$

with separation of variables,

but now $\phi$ is restricted to range $0 < \phi < \beta$.

We still have azimuthal symmetry,

but now, since we do not need solution to be finite for all $\phi \in [0, \pi]$, but only $\phi \in (0, \beta)$, we have more solutions to the \nabla^2 \phi equation, i.e. $\phi$ does not have to be integer. "Still need $b_\phi$ to be finite at $\phi = 0$.

See Jackson sec. 3.9 for details.