

Magnetostatics

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} \end{cases} \quad \text{Ampere's law (statics only!)}$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \frac{4\pi}{c} \vec{j}$$

$$\text{can write } \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

where by $\nabla^2 \vec{A}$ we mean $(\nabla^2 A_x) \hat{x} + (\nabla^2 A_y) \hat{y} + (\nabla^2 A_z) \hat{z}$

$\nabla^2 \vec{A}$ only has a single expression in Cartesian coords

If tried to write it in spherical coords, for example, one has

$$\begin{aligned} \nabla^2 \vec{A} &= \nabla^2 (A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}) \\ &= (\nabla^2 A_r) \hat{r} + A_r (\nabla^2 \hat{r}) + (\nabla^2 A_\theta) \hat{\theta} + A_\theta (\nabla^2 \hat{\theta}) \\ &\quad + (\nabla^2 A_\phi) \hat{\phi} + A_\phi (\nabla^2 \hat{\phi}) \end{aligned}$$

one must not forget to take the derivatives of $\hat{r}, \hat{\theta}, \hat{\phi}$ since they vary with position!

for example, $\hat{r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$

one could compute $\nabla^2 \hat{r}$ by applying ∇^2 in spherical coords to each piece and summing up. Get a mess!

If work in Coulomb gauge, with $\vec{\nabla} \cdot \vec{A} = 0$, then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \boxed{-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}} \quad \text{Poisson's equation!}$$

Many of the same methods used to solve for electrostatic ϕ can therefore be applied to solve for magnetostatic $\vec{A}$. But vector nature of $\vec{A}$ makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\vec{A}(\vec{r}) = \frac{1}{c} \int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad \text{three equations for } A_x, A_y, A_z!$$

for localized current sources $j(r) \rightarrow 0$ as $r \rightarrow \infty$

Multipole expansion - magnetic dipole moment

For a general treatment, analogous to how we did multipole expansion for electrostatics, one can use vector spherical harmonics - see Jackson Chpt 9.

Here we do a more straight forward approach, but only up to magnetic dipole term.

For $r \gg r'$ approx

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{(r^2 - 2\vec{r} \cdot \vec{r}' + r'^2)^{1/2}} = \frac{1}{r} \left[1 - \frac{2\vec{r} \cdot \vec{r}'}{r^2} + \left(\frac{r'}{r}\right)^2 \right]^{-1/2}$$

do Taylor series to 1st order in $(\frac{r'}{r})$ to get

$$\frac{1}{|\vec{r} - \vec{r}'|} \approx \frac{1}{r} \left\{ 1 + \frac{\vec{r} \cdot \vec{r}'}{r^2} + \dots \right\} = \frac{1}{r} + \frac{\vec{r} \cdot \vec{r}'}{r^3} + \dots$$

$$\vec{A}(\vec{r}) = \int \frac{d^3r'}{c} \frac{\vec{j}(\vec{r}')}{r} + \int \frac{d^3r'}{c} \vec{j}(\vec{r}') \frac{(\vec{r} \cdot \vec{r}')}{r^3} + \dots$$

Consider term ①

$$\int d^3r \vec{j}(\vec{r}) \quad \int d^3r (\vec{j} \cdot \vec{\nabla}) \vec{r} \quad \frac{\partial r_i}{\partial r_j} = \delta_{ij}$$

write $\int d^3r j_i(r) = \sum_{j=1}^3 \int d^3r j_j \frac{\partial r_i}{\partial r_j}$ integrate by parts

$$= \sum_j \left\{ \oint_S da j_j r_i - \int d^3r \frac{\partial j_j}{\partial r_j} r_i \right\}$$

↑
vanishes as $S \rightarrow \infty$ if
 $\vec{j}$ sufficiently localized
ie $\vec{j}(\vec{r}) \rightarrow 0$ sufficiently
fast as $r \rightarrow \infty$

↑
vanishes in
magnetostatics
where $\vec{\nabla} \cdot \vec{j} = 0$

So $\int d^3r \vec{j}(\vec{r}) = 0$ in magnetostatics
monopole term vanishes

term (2)

$$\int d^3r \vec{f}(\vec{r}) \vec{r} \quad \text{tensor}$$

Consider $\int d^3r \, j_i r_j = \sum_k \int d^3r \, j_k r_j \frac{\partial r_i}{\partial r_k}$ integrate by parts

$$= \sum_k \left\{ \oint_S j_k r_j r_i - \int d^3r \frac{\partial}{\partial r_k} (j_k r_j) r_i \right\}$$

$\uparrow$
vanishes as $S \rightarrow \infty$ if $\vec{f}$ sufficiently localized

$$= - \sum_k \int d^3r \left(\frac{\partial j_k}{\partial r_k} r_j r_i + j_k \frac{\partial r_j}{\partial r_k} r_i \right)$$

$\uparrow$ vanishes as $\vec{\nabla} \cdot \vec{f} = 0$ in magnetostatics $\uparrow = \delta_{jk}$

$$= - \int d^3r \, j_j r_i$$

So $\int d^3r \, j_i r_j = - \int d^3r \, j_j r_i$

$$= \frac{1}{2} \int d^3r (j_i r_j - j_j r_i)$$

So

$$\int d^3r' \, \vec{j}(\vec{r}') (\vec{r} \cdot \vec{r}') = \sum_j r_j \int d^3r' \, j_i(\vec{r}') r_j'$$

$$= \sum_j \frac{1}{2} \int d^3r' (j_i r_j r_j' - r_j j_j r_i')$$

$$= \frac{1}{2} \int d^3r' (j_i (\vec{r} \cdot \vec{r}') - r_i' (\vec{r} \cdot \vec{r}'))$$

use triple product rule

$$\vec{r} \times (\vec{r}' \times \vec{f}) = \vec{r}' (\vec{r} \cdot \vec{f}) - \vec{f} (\vec{r} \cdot \vec{r}')$$

to rewrite as

$$\int d^3r' \vec{f} (\vec{r} \cdot \vec{r}') = -\frac{1}{2} \vec{r} \times \left[\int d^3r' \vec{r}' \times \vec{f} (\vec{r}') \right]$$

define the magnetic dipole moment as

$$\boxed{\vec{m} = \frac{1}{2c} \int d^3r' \vec{r}' \times \vec{f} (\vec{r}')} \quad \leftarrow \text{definition}$$

In magnetic dipole approx (this is the lowest non-vanishing term)

$$\vec{A}(\vec{r}) = - \frac{\vec{r} \times \vec{m}}{r^2} = \frac{\vec{m} \times \vec{r}}{r^3} = \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{r}}{r^3} \right)$$

$$B_i = \epsilon_{ijk} \partial_j \epsilon_{klm} m_l \frac{r_m}{r^3}$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j m_l \frac{r_m}{r^3}$$

$$= m_l \partial_j \left(\frac{r_j}{r^3} \right) - m_j \partial_j \left(\frac{r_l}{r^3} \right)$$

$$= m_l \left[-4\pi \delta(\vec{r}) \right] - m_j \left[\frac{\delta_{lj}}{r^3} - \frac{3r_l}{r^4} \partial_j r \right]$$

$$= \underset{\substack{\uparrow \\ \text{for from source}}}{0} - \frac{m_l}{r^3} + \frac{3r_l}{r^4} \frac{r_j}{r} m_j$$

Levi-Civita symbol

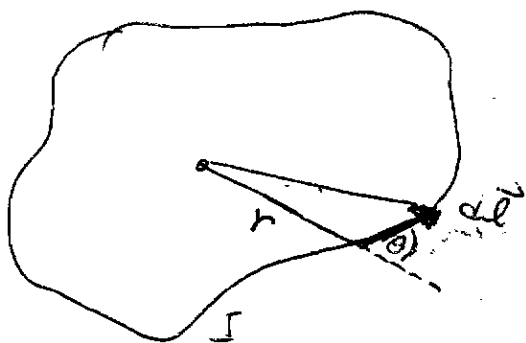
$$\epsilon_{ijk} = \begin{cases} 0, & \text{any two indices equal} \\ +1, & ijk \text{ even perm. of } 123 \\ -1, & ijk \text{ odd perm. of } 123 \end{cases}$$

$$\vec{B} = \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3}$$

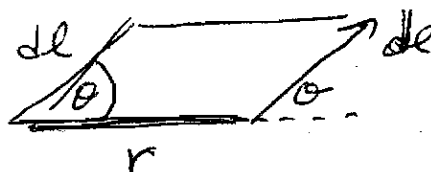
same form as $\vec{E}$ from electric dipole $\vec{p}$

For a current loop in a plane (any shape loop provided it is flat)

$$\vec{m} = \frac{1}{2c} \int d^3x \vec{r} \times \vec{j} = \frac{1}{2c} I \oint \vec{r} \times d\vec{l}$$



area of triangle is $\frac{1}{2} r dl \sin \theta$
 $= \frac{1}{2} |\vec{r} \times d\vec{l}|$



area of rectangle is $r dl \sin \theta$

$$\Rightarrow \vec{m} = \frac{1}{2} I (\text{area}) \hat{n}$$

$\uparrow$ area of loop $\nwarrow$ outward normal

(direction given by right hand rule with respect to direction of current)

magnetic dipole moment $\vec{m}$ is independent of location of origin.

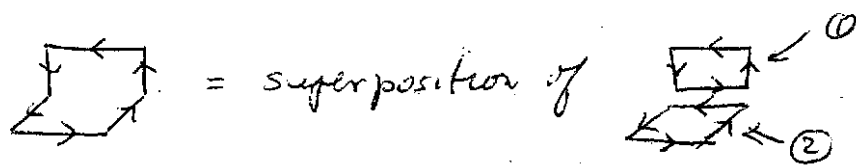
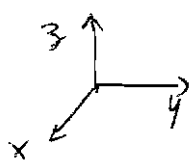
$$\vec{r}' = \vec{r} + \vec{d} \quad \text{new coord}$$

$$\begin{aligned} \vec{m}' &= \frac{1}{2c} \int d^3r' (\vec{r}' \times \vec{j}) = \frac{1}{2c} \int d^3r (\vec{r} + \vec{d}) \times \vec{j} \\ &= \frac{1}{2c} \int d^3r \vec{r} \times \vec{j} + \frac{1}{2c} \vec{d} \times \left[\int d^3r \vec{j} \right] \end{aligned}$$

$$\vec{m}' = \vec{m} + 0 \quad \text{as } \int d^3r \vec{j} = 0$$

for planar loop $\vec{m} = \frac{I}{c} a \hat{n}$ where $a = \text{area}$
 $\hat{n} = \text{outward normal}$

Can also apply to get $\vec{m}$ for piecewise planar loops



$$\vec{m} = \vec{m}_1 + \vec{m}_2$$

$$\vec{m}_1 = \frac{I}{c} a_1 \hat{x}$$

$$\vec{m}_2 = \frac{I}{c} a_2 \hat{z}$$

$$\Rightarrow \vec{m} = \frac{I}{c} (a_1 \hat{x} + a_2 \hat{z})$$