

Boundary value problems in magnetostatics

Scalar Magnetic Potential

Because of the vector character of the equation

$$-\nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}$$

and the fact that $\nabla^2 \vec{A}$ only has a convenient representation in Cartesian coordinates, many of the methods we used to solve the scalar $-\nabla^2 \phi = 4\pi\rho$ don't work so well for magnetostatics.

However, in situations where the current \vec{j} is confined to certain surfaces, we can make things much closer to the electrostatic case by using the trick of the scalar magnetic potential ϕ_M .

In regions where $\vec{j} = 0$, ie not on the certain surfaces, we have $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$. Since $\vec{\nabla} \times \vec{B} = 0$ in these regions, we can define a scalar potential ϕ_M such that

$$\vec{B} = -\vec{\nabla} \phi_M$$

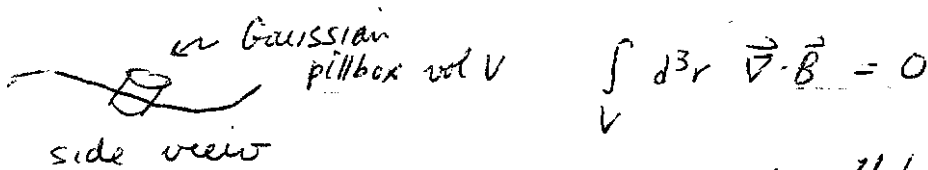
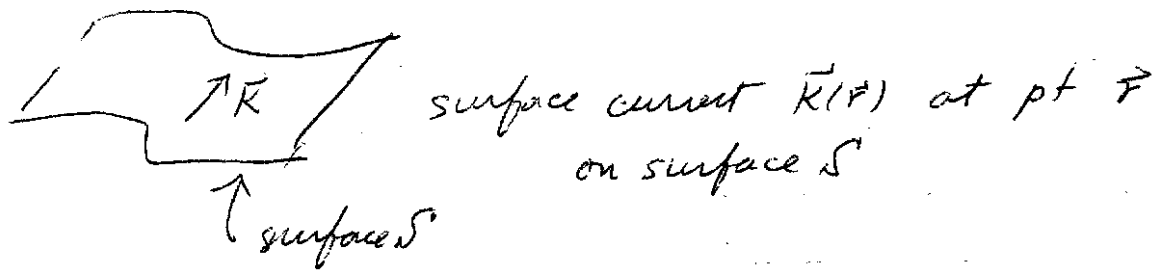
and then

$$\vec{\nabla} \cdot \vec{B} = -\nabla^2 \phi_M = 0$$

We can solve for ϕ_M as in electrostatics, and match solutions by applying appropriate boundary conditions on the current carrying surfaces.

Boundary Conditions at Sheet Current

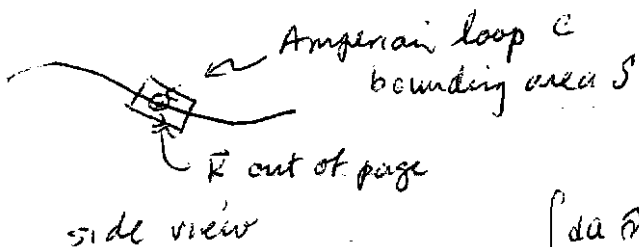
in magnetostatics $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$



top + bottom area of pill box is da
width of pill box $\rightarrow 0$

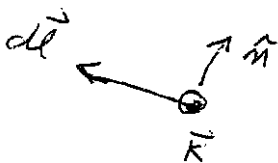
$$\Rightarrow \int_V d^3r \vec{\nabla} \cdot \vec{B} = \oint_S da \hat{n} \cdot \vec{B} = da (\vec{B}_{above} - \vec{B}_{below}) \cdot \hat{n} = 0$$

normal component of \vec{B} is continuous $(\vec{B}_{above} - \vec{B}_{below}) \cdot \hat{n} = 0$



$$\int_S da \hat{n} \cdot (\vec{\nabla} \times \vec{B}) = \oint_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{enclosed}$$

let width of loop $\rightarrow 0$, top + bottom sides $d\vec{l}$



$$(\vec{B}_{above} - \vec{B}_{below}) \cdot d\vec{l} = \frac{4\pi}{c} (\hat{n} \times d\vec{l}) \cdot \vec{K}$$

$$= \frac{4\pi}{c} (\vec{K} \times \hat{n}) \cdot d\vec{l}$$

tangential component of \vec{B} has discontinuous jump $\frac{4\pi}{c} \vec{K} \times \hat{n}$

Combine both results into

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

magnetic analog of $\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = 4\pi\sigma \hat{n}$

In terms of magnetic ~~vector~~^{scalar} potential Φ_M

$$-\vec{\nabla}\Phi_{M\text{above}} + \vec{\nabla}\Phi_{M\text{below}} = \frac{4\pi}{c} \vec{K} \times \hat{n}$$

Note: Φ_M is a calculational tool only
it does not have any direct physical
significance as does the electrostatic ϕ .
Electrostatic ϕ is related to work done
moving a charge $W_{12} = q[\phi(r_2) - \phi(r_1)]$
nothing similar for Φ_M .

(in fact magnetostatic magnetic forces do no work!)

$$\vec{F} = q \vec{v} \times \vec{B}$$
$$\Rightarrow \vec{F} \cdot \vec{v} = \frac{dW}{dt} = 0$$

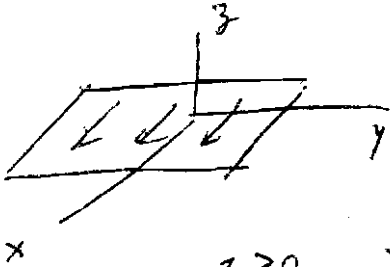
Note: We cannot apply argument ^{like} $\phi(r) - \phi(r') = \int_{r'}^r \vec{E} \cdot d\vec{l}$
 Φ_M is not necessarily continuous at surface ^{current}
Cannot do similar to electrostatics and use
 $\Phi_M(r_{\text{above}}) - \Phi_M(r_{\text{below}}) = - \int_{r_{\text{below}}}^{r_{\text{above}}} \vec{B} \cdot d\vec{l}$

Since Φ_M is not defined on the current sheet
itself, separating "above" from "below".

example

Flat infinite plane at $z=0$ with surface current

$$\vec{K} = K \hat{x}$$



$$z \geq 0, \quad \nabla^2 \phi_M^> = 0 \Rightarrow \phi_M^> = a^> - b_x^> x - b_y^> y - b_z^> z$$
$$z < 0, \quad \nabla^2 \phi_M^< = 0 \Rightarrow \phi_M^< = a^< - b_x^< x - b_y^< y - b_z^< z$$

$$z \geq 0, \quad \vec{B}^> = -\vec{\nabla} \phi_M^> = b_x^> \hat{x} + b_y^> \hat{y} + b_z^> \hat{z}$$
$$z < 0, \quad \vec{B}^< = -\vec{\nabla} \phi_M^< = b_x^< \hat{x} + b_y^< \hat{y} + b_z^< \hat{z}$$

$$\text{at } z=0 \quad \vec{B}^> - \vec{B}^< = (b_x^> - b_x^<) \hat{x} + (b_y^> - b_y^<) \hat{y} + (b_z^> - b_z^<) \hat{z}$$
$$= \frac{4\pi}{c} \vec{K} \times \hat{m} = \frac{4\pi}{c} K (\hat{x} \times \hat{z}) = -\frac{4\pi}{c} K \hat{y}$$

$$\Rightarrow b_x^> = b_x^< \equiv b_{x0}, \quad b_z^> = b_z^< \equiv b_{z0}, \quad b_y^> - b_y^< = -\frac{4\pi}{c} K$$

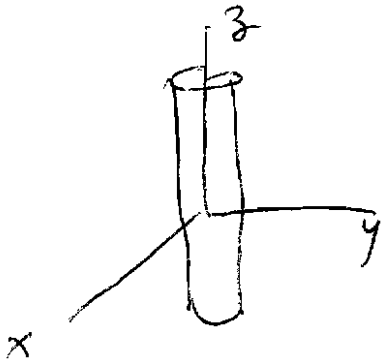
$$\text{define } \left. \begin{aligned} b_y^> &= b_{y0} + \delta b_y \\ b_y^< &= b_{y0} - \delta b_y \end{aligned} \right\} \delta b_y = -\frac{2\pi}{c} K$$

$$\Rightarrow \vec{B}^> = \vec{B}_0 - \frac{2\pi}{c} K \hat{y} \quad \vec{B}_0 = b_{x0} \hat{x} + b_{y0} \hat{y} + b_{z0} \hat{z}$$
$$\vec{B}^< = \vec{B}_0 + \frac{2\pi}{c} K \hat{y}$$

if \vec{K} is the only source of magnetic field then $\vec{B}_0 = 0$

$$\vec{B} = \begin{cases} -\frac{2\pi}{c} K \hat{y} & z > 0 \\ \frac{2\pi}{c} K \hat{y} & z < 0 \end{cases}$$

example current carrying infinite cylinder radius R



- (i) $\vec{K} = K \hat{z}$ wire with surface current
 (ii) $\vec{K} = K \hat{\phi}$ solenoid

(i) $\vec{K} = K \hat{z}$

$2\pi R K = I$ total current
 "guess" + show it is correct

$r > R$ $\Phi_M = -\frac{4\pi R K \phi}{c}$
 $r < R$ $\Phi_M = 0$
 magnetic scalar potential $\nabla^2 \Phi_M = 0$

$r > R$ $\vec{B} = -\vec{\nabla} \Phi_M = -\frac{1}{r} \frac{\partial \Phi_M}{\partial \phi} \hat{\phi} = \frac{4\pi R K}{c r} \hat{\phi} = \frac{2I}{c r} \hat{\phi}$
 $r < R$ $\vec{B} = 0$
 ← further result from Ampere

$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \frac{2I}{cR} \hat{\phi} = \frac{4\pi K}{c} \frac{R}{R} \hat{\phi} = \frac{4\pi \vec{K}}{c} \times \hat{n}$
 where $\hat{n} = \hat{r}$
 as $\hat{z} \times \hat{r} = \hat{\phi}$

Note: $\Phi_M = -\frac{4\pi R K \phi}{c}$ is not single valued!
 would not have found this using expansion of separation of coords in polar coords

(ii) $\vec{K} = K \hat{\phi}$

$r > R$ $\Phi_M = -B_1 z$
 $r < R$ $\Phi_M = -B_2 z$
 $r > R$ $\vec{B} = -\vec{\nabla} \Phi_M = B_1 \hat{z}$
 $r < R$ $\vec{B} = -\vec{\nabla} \Phi_M = B_2 \hat{z}$
 $\left. \begin{array}{l} \Phi_M = -B_1 z \\ \Phi_M = -B_2 z \end{array} \right\} \nabla^2 \Phi_M = 0$

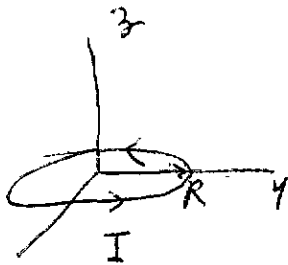
$$\begin{aligned}\vec{B}_{\text{above}} - \vec{B}_{\text{below}} &= (B_1 - B_2) \hat{z} = \frac{4\pi}{c} \vec{K} \times \hat{n} \\ &= \frac{4\pi}{c} K (\hat{\phi} \times \hat{r}) \\ &= -\frac{4\pi}{c} K \hat{z}\end{aligned}$$

If current in solenoid is only source of \vec{B} . Then expect $B_1 = 0$.

$$\Rightarrow \boxed{\vec{B}_2 = \frac{4\pi}{c} K \hat{z}} \text{ familiar result.}$$

ax axis

circular current loop in xy plane
radius R



for $r > R$, $\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B} = -\vec{\nabla} \phi_M$
where $\nabla^2 \phi_M = 0$.

Try Legendre polynomial expansion for ϕ_M

$$\phi_M = \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta) \quad (A_{\ell} \text{ terms vanish as want } B \rightarrow 0 \text{ as } r \rightarrow \infty)$$

$$\vec{B} = -\vec{\nabla} \phi_M = -\frac{\partial \phi_M}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi_M}{\partial \theta} \hat{\theta}$$

$$= \sum_{\ell} \left[\frac{(\ell+1) B_{\ell}}{r^{\ell+2}} P_{\ell}(\cos \theta) \hat{r} - \frac{B_{\ell}}{r^{\ell+2}} \frac{\partial P_{\ell}(\cos \theta)}{\partial \theta} \hat{\theta} \right]$$

write $\frac{\partial P_{\ell}}{\partial \theta} = \frac{\partial P_{\ell}}{\partial x} \frac{\partial x}{\partial \theta} = -\frac{\partial P_{\ell}}{\partial x} \sin \theta \quad x = \cos \theta$
 $\equiv -P'_{\ell} \sin \theta$

$$\vec{B} = \sum_{\ell} \left[\frac{(\ell+1) B_{\ell}}{r^{\ell+2}} P_{\ell}(\cos \theta) \hat{r} + \frac{B_{\ell}}{r^{\ell+2}} \sin \theta P'_{\ell}(\cos \theta) \hat{\theta} \right]$$

To determine the B_{ℓ} we compare with exact solution along \hat{z} axis

$$\vec{B}(z\hat{z}) = \sum_{\ell} \frac{(\ell+1) B_{\ell}}{r^{\ell+2}} \hat{r} = \sum_{\ell} \frac{(\ell+1) B_{\ell}}{z^{\ell+2}} \hat{z}$$

since $P_{\ell}(1) = 1$, $\sin(0) = 0$ and $P'_{\ell}(1)$ finite, $\hat{r} = \hat{z}$ when $\theta = 0$

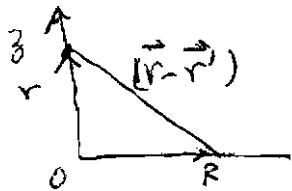
exact solution on \hat{z} axis:

$$\vec{A} = \int \frac{d^3 r'}{c} \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A} = \int \frac{d^3 r'}{c} \vec{\nabla} \times \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\vec{B} = - \int \frac{d^3 r'}{c} \vec{j}(\vec{r}') \times \vec{\nabla} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$$

$$\vec{B} = \int \frac{d^3 r'}{c} \vec{j}(\vec{r}') \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad \text{Biot-Savart Law for magnetostatics}$$

For our loop



$$\vec{B}(z) = \int_0^{2\pi} d\phi \frac{R}{c} I \hat{\phi} \times \frac{[-R \hat{r} + z \hat{z}]}{(z^2 + R^2)^{3/2}} \quad \swarrow \text{polar radial vector}$$

$$\hat{r} \times \hat{\phi} = \hat{z}$$

$$= \int_0^{2\pi} d\phi \frac{R(I R) \hat{z}}{c (z^2 + R^2)^{3/2}}$$

$\hat{\phi} \times \hat{z}$ term integrates to zero

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c (z^2 + R^2)^{3/2}}$$

to match Legendre polynomial expansion, do Taylor series expansion of above

$$\vec{B}(z) = \frac{2\pi R^2 I \hat{z}}{c z^3} \frac{1}{\left(1 + \left(\frac{R}{z}\right)^2\right)^{3/2}} = \frac{2\pi R^2 I \hat{z}}{c z^3} \left\{ 1 - \frac{3}{2} \left(\frac{R}{z}\right)^2 + \dots \right\}$$

$$= \frac{2\pi R^2 I \hat{z}}{c} \left\{ \frac{1}{z^3} - \frac{3}{2} \frac{R^2}{z^5} + \dots \right\}$$

$$= \left\{ \frac{B_0}{z^2} + \frac{2B_1}{z^3} + \frac{3B_2}{z^4} + \frac{4B_3}{z^5} + \dots \right\} \hat{z}$$

$$\Rightarrow B_0 = 0, \quad B_1 = \frac{\pi R^2 I}{c}, \quad B_2 = 0, \quad B_3 = -\frac{3\pi R^2 I R^2}{4c}$$

So to order $L=3$

$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 P_1(\cos\theta) \hat{r} + \sin\theta P_1'(\cos\theta) \hat{\theta}}{r^3} - \left[\frac{3R^2 P_3(\cos\theta) \hat{r} + \frac{3}{4} R^2 \sin\theta P_3'(\cos\theta) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$$P_1(x) = x \Rightarrow P_1'(x) = 1$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \frac{1}{2}(15x^2 - 3)$$

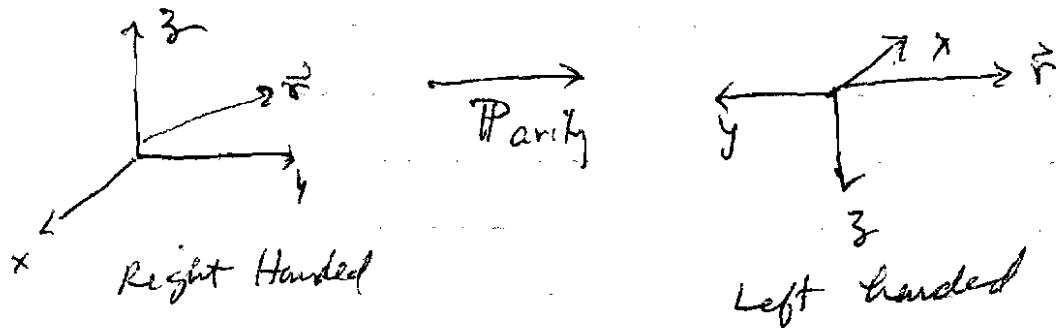
$$\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 \cos\theta \hat{r} + \sin\theta \hat{\theta}}{r^3} - \left[\frac{\frac{3}{2} R^2 (5 \cos^3\theta - 3 \cos\theta) \hat{r} + \frac{3}{8} R^2 \sin\theta (15 \cos^2\theta - 3) \hat{\theta}}{r^5} \right] + \dots \right\}$$

$\frac{\pi R^2 I}{c} = m$ is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic ^{octapole} ~~quadrupole~~ term. Could easily get higher order terms by this method.

Compare our result above to Jackson (5.40)

Symmetry under parity transformation vector vs. pseudo vector



$$\vec{r} = (x, y, z) \rightarrow (-x, -y, -z)$$

$P(\vec{r}) = -\vec{r}$ position \vec{r} is odd under parity

Any vector-like quantity that is odd under P is a vector.

examples of vectors

position \vec{r}
 velocity $\vec{v} = \frac{d\vec{r}}{dt}$ since \vec{r} is vector and t is scalar
 acceleration $\vec{a} = \frac{d\vec{v}}{dt}$ $P(t) = t$

Force $\vec{F} = m\vec{a}$ since \vec{a} is vector and m is scalar
 momentum $\vec{p} = m\vec{v}$ since \vec{v} is vector and m is scalar

electric field $\vec{F} = q\vec{E}$ since \vec{F} is vector and q is scalar

current $\vec{j} = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i(t))$ $P(q) = q$

any vector-like quantity that is even under \mathcal{P} is a pseudovector

angular momentum $\vec{L} = \vec{r} \times \vec{p}$

since $\vec{r} \rightarrow -\vec{r}$ and $\vec{p} \rightarrow \vec{p}$,
 $\vec{L} \rightarrow \vec{L}$ under \mathcal{P}

\vec{L} is even under \mathcal{P}

magnetic field $\vec{F} = g \vec{v} \times \vec{B}$

since \vec{F} and \vec{v} are vectors and g is scalar, \vec{B} must be pseudovector.

cross product of any two vectors is a pseudovector
 " " " vector and pseudovector is a vector

when solving for \vec{E} , it can only be made up of vectors that exist in the problem

when solving for \vec{B} , it can only be made up of pseudovectors that exist in the problem

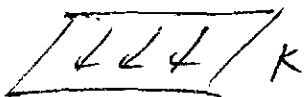
ex charged plane



only directions in problem is normal \hat{m}
 \hat{m} is a vector

$$\vec{E} \propto \hat{m}$$

surface current



only directions are the vectors \hat{m} and \vec{K} . But \vec{B} can only be made of pseudovectors

$$\Rightarrow \vec{B} \propto (\vec{K} \times \hat{m})$$