Maxwell's equations apply exactly to the free microscopic electric and magnetic fields that arise from all charges ad currents.

\[
\mathbf{E} \cdot \mathbf{b} = 0 \quad \nabla \times \mathbf{E} + \frac{1}{\varepsilon} \frac{\partial \mathbf{b}}{\partial t} = 0
\]

\[
\mathbf{B} \cdot \mathbf{E} = 4\pi \rho \quad \nabla \times \mathbf{B} = \frac{\mu_0}{\varepsilon} \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{D}}{\partial t}
\]

where \( \mathbf{E} \) and \( \mathbf{B} \) are microscopic fields from total charge density \( \rho \) and current density \( \mathbf{j}_0 \).

However, in most problems involving macroscopic objects, if we took \( \rho \) and \( \mathbf{j}_0 \) to describe charge and current of each individual atom in a material, then they, and the resulting \( \mathbf{E} \) and \( \mathbf{B} \), would be enormously complicated functions varying rapidly over distances \( \sim 10^{-8} \) cm and times \( \sim 10^{-10} \) sec.

In classical E&M we are generally concerned with phenomena that vary extremely slowly compared to these length and time scales.
Rather than worry about the microscopic details of $\mathbf{J}$ and $\mathbf{P}$ or $\mathbf{E}$ and $\mathbf{B}$ we want to describe phenomena in terms of averaged smooth Lorentz-force averaged quantities that are smoothly varying at the atomic scale. This results in what are known as the **macroscopic Maxwell equations**

**Dielectric Materials**

A dielectric material is an insulator. Electrons are bound to the ionic cores of the atoms. When no electric field is present, the averaged $\mathbf{P}$ in the dielectric vanishes. One might therefore think that electric dynamics in a dielectric is just due to whatever "extra" or "free" charge is added to the dielectric. However, this is not true due to the phenomenon of "polarization".

\[ \mathbf{E} = 0 \quad \text{and} \quad \mathbf{E} \neq 0 \]

- **Electron cloud** centered on ionic core
- Dipole moment vanishes

- **Electron cloud** and ionic core displaced $\mathbf{d} \propto \mathbf{E}$
- Atom is "polarized" has dipole moment $\mathbf{P} = q \mathbf{d} \propto q \mathbf{E}$
Polarization density \( \mathbf{P}(\mathbf{r}) = \sum_i \mathbf{P}_i \delta(\mathbf{r} - \mathbf{r}_i) \)

* Dipole moment of atom at position \( \mathbf{r}_i \).

Polarization density \( \mathbf{P} \) can give rise to regions of net charge - sometimes called "bound charge".

**Example**

\[ \mathbf{P} \rightarrow \text{in terms of averaged charge} \]

\[ \mathbf{E} = 0 \quad \text{uniform} \mathbf{E} \rightarrow \mathbf{E} > 0 \]

\[ \text{neutral} \quad (+) \quad (-) \quad \text{surface charge builds up} \]

For a non-uniform \( \mathbf{E} \), atoms are more strongly polarized where \( E \) is largest.

\[ \mathbf{E} \rightarrow \]

\[ \text{net positive charge region} \]

For uniform \( \mathbf{P} \), build up surface charge \( \sigma_b \).

For non-uniform \( \mathbf{P} \), also can build up vol charge density \( \sigma_b \).
We now carry out the averaging explicitly to see how such polarization enters the macroscopic Maxwell equations (Jackson 6.6).

Define spatially averaged quantities by

$$\langle F(\vec{r},t) \rangle = \int d^3r' f(\vec{r}') F(\vec{r} - \vec{r}',t)$$

where $f(\vec{r})$ vanishes for $|\vec{r}|$ large on microscopic length scales, but short on macroscopic length scales. $\int f(\vec{r})$ normalized to unity $\int d^3r f(\vec{r}) = 1$.

Other details of $f(\vec{r})$ are not too important, as long as $f(\vec{r})$ is a smooth function of $\vec{r}$.

\[\begin{array}{c}
\text{want } f \approx 1 \text{ for } r < R \\
\text{for } r > R \\
\text{where } R \text{ is length scale midway between micro and macro}
\end{array}\]

\[\begin{array}{c}
\frac{\partial}{\partial r} \langle F(\vec{r},t) \rangle = \int d^3r' f(\vec{r}') \frac{\partial F(\vec{r} - \vec{r}')}{\partial r} = \langle \frac{\partial F}{\partial r} \rangle \\
2 \frac{\partial}{\partial t} \langle F(\vec{r},t) \rangle = \langle \frac{\partial^2 F}{\partial t^2} \rangle
\end{array}\]

Define the macroscopic fields

$$\vec{E}(\vec{r},t) = \langle \vec{E}(\vec{r},t) \rangle$$
$$\vec{B}(\vec{r},t) = \langle \vec{B}(\vec{r},t) \rangle$$
\[ \nabla \cdot \mathbf{b} = 0 \implies < \nabla \cdot \mathbf{b} > = 0 \]
\[ \implies \nabla \cdot < \mathbf{b} > = 0 \]
\[ \implies \nabla \times < \mathbf{b} > = 0 \]

\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \implies \nabla \times < \mathbf{E} > + \frac{\partial < \mathbf{B} >}{\partial t} = 0 \]
\[ \implies \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \]

Runaway Maxwell equ. upon averaging, become

\[ \nabla \cdot \mathbf{E} = 4\pi < \rho_0 > \]
\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} < \mathbf{J}_0 > + \frac{\partial \mathbf{E}}{\partial t} \]

Consider \( < \rho_0 > \)

\[ \rho_0 = \sum_i q_i \delta (\mathbf{r} - \mathbf{r}_i(t)) \] sum over all charges

Consider dividing the charge into "free" charges and "bound" charges, where the latter are associated with the molecules that make up the dielectric

\[ \rho_{\text{free}} = \sum_{i \text{free}} q_i \delta (\mathbf{r} - \mathbf{r}_i(t)) \] sum over only free charges

\[ \rho_{\text{bound}} = \sum_n \rho_n (\mathbf{r}, t) \] charge distribution of molecule \( n \)

\[ \rho_n = \sum_{c} q_i \delta (\mathbf{r} - \mathbf{r}_i(t)) \] sum over charges in molecule \( n \)
\[
\langle \mathbf{r}_n(\mathbf{r}, t) \rangle = \int d^3 r' \, \mathbf{f}(r', t) \, \mathbf{r}_n(\mathbf{r}' - \mathbf{r}_c(t))
\]

\[
= \sum_{i} q_i \int d^3 r' \, \mathbf{f}(r') \cdot \mathbf{r}_c(t) - \mathbf{r}_c(t)
\]

\[
= \sum_{i} q_i \, \mathbf{f}(\mathbf{r} - \mathbf{r}_c(t))
\]

Write \( \mathbf{r}_c(t) = \mathbf{r}_n(t) + \mathbf{r}_{nc}(t) \)

\[
\langle \mathbf{r}_n(\mathbf{r}, t) \rangle = \sum_{i} q_i \, \mathbf{f}(\mathbf{r} - \mathbf{r}_n - \mathbf{r}_{nc})
\]

Since the \( \mathbf{r}_{nc} \) are all on an atomic length scale, and \( \mathbf{f} \) is slowly varying on the length scale, we can expand

\[
\langle \mathbf{r}_n(\mathbf{r}, t) \rangle = \sum_{i} q_i \left[ \mathbf{f}(\mathbf{r} - \mathbf{r}_n) - \left( \nabla \mathbf{f}(\mathbf{r} - \mathbf{r}_n) \right) \cdot \mathbf{r}_{nc} + \frac{1}{2} \sum_{\alpha, \beta} \frac{\partial^2 \mathbf{f}(\mathbf{r} - \mathbf{r}_n)}{\partial r_\alpha \partial r_\beta} \mathbf{r}_{nc, \alpha} (\mathbf{r}_{nc, \beta} - \mathbf{r}_{nc}) \right] + \ldots
\]

\[
= \mathbf{f}(\mathbf{r} - \mathbf{r}_n) \sum_{i} q_i \Delta \mathbf{r}_n
\]

\[
- \left( \nabla \mathbf{f}(\mathbf{r} - \mathbf{r}_n) \right) \cdot \sum_{i} q_i \mathbf{r}_{nc}
\]

\[
+ \frac{3}{2} \sum_{\alpha, \beta} \left( \frac{\partial^2 \mathbf{f}(\mathbf{r} - \mathbf{r}_n)}{\partial r_\alpha \partial r_\beta} \right) \sum_{i} q_i \mathbf{r}_{nc, \alpha} (\mathbf{r}_{nc, \beta} - \mathbf{r}_{nc})
\]
Define $q_n = \sum_{i \in n} q_i$ total charge molecule $n$

$\vec{p}_n = \sum_{i \in n} q_i \vec{r}_i$ dipole moment about center of mass of molecule $n$

$\vec{Q}_n = \sum_{i \in n} \frac{3}{2} q_i \vec{r}_i \times \vec{r}_i$ quadrupole moment about center of mass of molecule $n$

(prime on $\vec{Q}$ was defined here is a little different from that of multipole exp)

$$<\phi_n(\vec{r},t)> = q_n f(\vec{r}-\vec{r}_n) - \vec{p}_n \cdot \vec{\nabla} f(\vec{r}-\vec{r}_n)$$

$$+ \frac{1}{6} \sum_{\alpha \beta} (\frac{\partial^2 f(\vec{r}-\vec{r}_n)}{\partial r_\alpha \partial r_\beta})$$

Now use $<\delta(\vec{r}-\vec{r}_n)> = f(\vec{r}-\vec{r}_n)$ by definition of autagony

$$\Rightarrow <\phi_n(\vec{r},t)> = <q_n \delta(\vec{r}-\vec{r}_n)>$$

$$- \vec{\nabla} \cdot <\vec{p}_n \delta(\vec{r}-\vec{r}_n)>$$

$$+ \frac{1}{6} \sum_{\alpha \beta} (\frac{\partial^2}{\partial r_\alpha \partial r_\beta}) <(Q_n^\prime)_{\alpha \beta} \delta(\vec{r}-\vec{r}_n)>$$

Now

$$<\phi_{\text{bound}}(\vec{r},t)> = \sum_n <\phi_n(\vec{r},t)>$$

$$= <\sum_n q_n \delta(\vec{r}-\vec{r}_n)> - \vec{\nabla} \cdot <\sum_n \vec{p}_n \delta(\vec{r}-\vec{r}_n)>$$

$$+ \frac{1}{6} \sum_{\alpha \beta} (\frac{\partial^2}{\partial r_\alpha \partial r_\beta}) <\sum_n (Q_n^\prime)_{\alpha \beta} \delta(\vec{r}-\vec{r}_n)>$$
Define \( \mathbf{P}(r,t) = \left\langle \sum_n P_n \delta(r-r_n) \right\rangle \) average polarized density
\[
\mathbf{Q}'(r,t) = \frac{1}{6} \left\langle \sum_n Q'_n \delta(r-r_n) \right\rangle \text{ average quadrupole density}
\]

\[
\left\langle \mathbf{P}_{\text{bound}} \right\rangle = \sum_n \left\langle q_n \delta(r-r_n) \right\rangle - \nabla \cdot \mathbf{P} + \sum_{\alpha \beta} \frac{2}{\alpha \beta} \nabla \cdot \mathbf{Q}_{\alpha \beta}
\]

Define the macroscopic charge density
\[
\rho = \left\langle \sum_{i \text{ free}} \delta(r-r_i) \right\rangle + \left\langle \sum_{\text{molecule}} q_n \delta(r-r_n) \right\rangle
\]

Then
\[
\nabla \cdot \mathbf{E} = \left\langle \rho_0 \right\rangle = 4\pi \left[ \rho - \nabla \cdot \mathbf{P} + \sum_{\alpha \beta} \frac{2}{\alpha \beta} \nabla \cdot \mathbf{Q}_{\alpha \beta} \right]
\]
\[
\sum_{\alpha \beta} \frac{2}{\alpha \beta} \left[ E_\alpha + 4\pi P_\alpha - 4\pi \frac{2}{\beta} \nabla \cdot \mathbf{Q}_{\alpha \beta} \right] = 4\pi \rho
\]

Define electric displacement vector
\[
\mathbf{D}_\alpha = E_\alpha + 4\pi P_\alpha - 4\pi \frac{2}{\beta} \nabla \cdot \mathbf{Q}_{\alpha \beta}
\]
then
\[
\nabla \cdot \mathbf{D} = 4\pi \rho
\]

In most materials, the quadrupole and higher terms are negligible and we can take
\[
\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}
\]
Why quadrupole and higher order terms can generally be ignored:

Let \( a_0 \) be the length scale that characterizes the size of a molecule in the dielectric.
Let \( l \) be the typical spacing between molecules.
Let \( L \) be the length scale of the spatial averaging function \( f(r) \). \( L \gg a_0 \)

dipole moment \( p = a_0 \)

\[ \Rightarrow \text{polarization} \quad \mathbf{P} = \frac{a_0}{L^3} \]

\[ \nabla \cdot \mathbf{P} \sim \left( \frac{a_0}{L} \right) \frac{1}{L^3} \quad \text{since P cannot vary on length scale short than the averaging length L} \]

quadrupole moment

\[ Q \sim a_0^2 \]

quadrupole density \( \mathbf{Q} \sim \frac{a_0^2}{L^3} \)

\[ \frac{\partial^2 Q}{\partial x \partial y} \sim \left( \frac{a_0}{L} \right)^2 \frac{1}{L^3} \]

Each higher moment gives extra factor \( a_0 \).
Each higher derivative gives extra factor \( \frac{1}{L} \)

so quadrupole is smaller than dipole term by factor \( \left( \frac{a_0}{L} \right) \ll 1 \) Higher terms smaller by additional factors of \( \left( \frac{a_0}{L} \right) \)
For insulators one generally has $q_n = 0$
molecules are neutral.
In this case the macroscopic $\mathbf{\dot{F}}$ is just the
free charge $\mathbf{\dot{F}} = \langle \mathbf{\dot{F}}_{\text{free}} \rangle$.
And the bound charge $\mathbf{\dot{F}}$ is just
\[
\langle \mathbf{\dot{F}}_{\text{bound}} \rangle = - \nabla \alpha \mathbf{P}
\]
At a surface of a dielectric
\[
\begin{align*}
\text{area} & \quad \mathbf{n} \quad \text{Gaussian pillbox } V \\
\text{width } l & \to 0
\end{align*}
\]
\[
- \int d^3r \ \nabla \alpha \mathbf{P} = - \int \text{area} \ \mathbf{n} \cdot \mathbf{P}
\]
contrib from sides $\to 0$ as $l \to 0$
contrib from outside surface $= 0$
as $P = 0$ outside
\[
= \mathbf{n} \cdot \mathbf{P} \text{ area}
\]
only contrib from inside surface
\[
= \int \text{area}_{\text{bound}}
\]
($\mathbf{n}$ is outward normal)
\[
as \ l \to 0, \ \int d^3r_{\text{bound}} \to \int \text{area} \ \sigma_{\text{bound}} = \text{da} \ \sigma_{\text{bound}} \ \text{surface charge}
\]
\[
\Rightarrow \ \sigma_{\text{bound}} = \mathbf{n} \cdot \mathbf{P}
\]
at surface of dielectric
\[\square\]