Magnetic Materials

Circulating currents on atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material.

Sources of circulating atomic currents:
1. Intrinsic angular momentum of electrons, i.e. "electron spin" - can add up and give a net angular momentum to atom.
2. Orbital angular momentum of electrons - can add up to give net angular momentum of atom.

(1) + (2) ⇒ atoms can have a net magnetic dipole moment. When \( \mathbf{B} = 0 \), these atomic moments are generally in random orientations (exception is a ferromagnet where moments can align even if \( \mathbf{B} = 0 \)). When applied \( \mathbf{B} \), the moments tend to align parallel to \( \mathbf{B} \) giving a net magnetization density \( \mathbf{M} \times \mathbf{B} \). This is a paramagnetic effect.

But there is also a diamagnetic effect from orbital angular momentum (exists even if total angular momentum of electrons is zero, i.e. exists for atoms with zero net dipole moment).
\[ \vec{B} \]

applying \( \vec{B} \) to orbiting electron speeds up orbiting electron

its orbital velocity increased angular momentum of negatively charged electron

gives change in dipole moment \( \Delta \vec{m} = \vec{B} \)

\[ \vec{m}_0 \]

orbital dipole

\[ \vec{e} \]

applying \( \vec{B} \) to orbiting electron slows down its orbital velocity. Net result is again that \( \Delta \vec{m} = \vec{B} \)

No matter which way electron orbits with respect to \( \vec{B} \), result is a decrease in magnetic moment, so \( \Delta \vec{m} \propto \vec{B} \) that \( \Delta \vec{m} \) is opposite to \( \vec{B} \) is called diamagnetism

Model atomic magnetic moments as small current loops. When loops get oriented, as there is non zero average magnetization density

\[ \vec{M}(\vec{r}) = \sum \vec{m}_i \delta(\vec{r} - \vec{r}_i) \]

Then net effect is to have a current flowing around the system. The current gives rise to magnetic fields
alleged atomic moments in a uniform applied \( \mathbf{B} \)

in interior, currents in opposite directions cancel also \( \mathbf{J} = 0 \) inside but net circulation of current around boundary of material \( \Rightarrow \) surface current \( \mathbf{J}_{\text{bound}} \)

If \( \mathbf{B} \) is not uniform, then \( \mathbf{M} \) is not uniform
Can create finite current density \( \mathbf{J} \) in interior as well as surface currents

Now current in interior do not cancel, net current \( \mathbf{J}_{\text{bound}} \) in interior

\( \mathbf{B} \) strong \( \mathbf{B} \) weak

\( \mathbf{B} \) out of page \( \Rightarrow \) \( \mathbf{M} \) out of page
\( \mathbf{M} \) varies in direction \( \perp \) direction \( \mathbf{M} \)
\( \Rightarrow \) \( \nabla \times \mathbf{M} \neq 0 \) gives \( \mathbf{J}_{\text{bound}} \)
Average current

\[ \langle \vec{I}_{e} \rangle = \sum_{i \in \text{free}} \langle \vec{I}_{e} (\vec{r} - \vec{r}_{i} (t)) \rangle + \sum_{n} \langle \vec{I}_{n} \rangle \]

current from free charges

\[ \langle \vec{f}_{n} (\vec{r}_{i}, t) \rangle = \sum_{i \in n} \frac{\partial}{\partial t} \left( \vec{r}_{n} + \vec{v}_{n} \right) \langle \nabla \cdot (\vec{r} - \vec{r}_{n}(t) - \vec{r}_{n}(t)) \rangle \]

\[ = \sum_{i \in n} \frac{\partial}{\partial t} \left( \vec{r}_{n} + \vec{v}_{n} \right) f_{n} (\vec{r} - \vec{r}_{n}(t) - \vec{r}_{n}(t)) \]

velocity of molecule \( n \)

\[ \vec{v}_{n} = \frac{d\vec{r}_{n}}{dt}, \quad \vec{v}_{n} = \frac{d\vec{r}_{n}}{dt}, \quad \text{position of charge} \]

\[ \vec{r}_{n} = \text{CM of molecule } n \]

\[ \text{wrt. CM} \]

As with \( \langle \vec{f}_{o} \rangle \), we can expand in \( \vec{r}_{n} \)

\[ \langle \vec{f}_{n} \rangle = \sum_{i \in n} \frac{\partial}{\partial t} \left( \vec{r}_{n} + \vec{v}_{n} \right) f_{n} (\vec{r} - \vec{r}_{n}) - \vec{r}_{n} \cdot \nabla f_{n} (\vec{r} - \vec{r}_{n}) \]

we will keep only the first two terms in the expansion.
The various terms we have to consider are

\[ \sum_{i \in n} g_i \bar{v}_i \cdot f(r - \bar{r}_n) \]

\[ \sum_{i \in n} g_i \vec{v}_i \cdot f(r - \bar{r}_n) \]

\[ - \sum_{i \in n} \bar{g}_i \bar{v}_n \left[ \frac{\bar{v}_n \cdot \vec{v} (r - \bar{r}_n)}{r} \right] \]

\[ - \sum_{i \in n} \bar{g}_i \vec{v}_i \left[ -\frac{\vec{v} (r - \bar{r}_n) \cdot \bar{v}_n}{r^2} \right] \]

\[ \bar{v}_n + \left( r - \bar{r}_n \right) \sum_{i \in n} g_i = \bar{g}_n \vec{v}_n \cdot f(r - \bar{r}_n) \]

this is just current of molecule as if it were a point charge \( \bar{g}_n \). For a neutral molecule, \( \bar{g}_n = 0 \) as the term vanishes.

\[ \frac{d}{dt} \left( \bar{p}_n \delta (r - \bar{r}_n) \right) = \frac{\partial}{\partial t} \left( \sum_{i \in n} g_i \bar{v}_i \cdot f(r - \bar{r}_n) \right) \]

\[ \sum_{i \in n} g_i \vec{v}_i \cdot f(r - \bar{r}_n) \]

\[ + \sum_{i \in n} \bar{g}_i \bar{v}_n \left[ -\frac{\vec{v} (r - \bar{r}_n) \cdot \bar{v}_n}{r^2} \right] \]

So for \( \mathbb{R} \), \( \sum_{i \in n} g_i \bar{v}_i \cdot f(r - \bar{r}_n) \)

\[ = \frac{d}{dt} \left( \bar{p}_n \delta (r - \bar{r}_n) \right) \]

\[ + \left[ \bar{v}_n \cdot \vec{v} (r - \bar{r}_n) \right] \bar{p}_n \]
\( \mathbf{r} = \sum q_i \mathbf{r}_{ni} \) where \( \mathbf{r}_{ni} \) is the position of the center of mass of molecule \( i \) with respect to its own center of mass.

We have seen the tensor \( \sum q_i \mathbf{r}_{ni} \mathbf{v}_{ni} \) before when we considered the magnetic dipole moment.

\[
\mathbf{J}(\mathbf{r}) = \sum q_i \mathbf{r}_{ni} \mathbf{v}_{ni}
\]

is current density with respect to center of mass of molecule.

We had

\[
\int d^3 r \, \mathbf{J} = -\int d^3 r \, \mathbf{E} \cdot \mathbf{r} + \int d^3 r \, \frac{\partial}{\partial t} \mathbf{r}
\]

in states, \( \nabla \cdot \mathbf{J} = 0 \)

in general

\[
\nabla \cdot \mathbf{J} = -\frac{\partial}{\partial t} \mathbf{r}
\]

For \( \mathbf{r} \neq 0 \) we have

\[
\int d^3 r \, \mathbf{J} \cdot \mathbf{r} = -\int d^3 r \, \mathbf{E} \cdot \mathbf{r} + \int d^3 r \, \frac{\partial}{\partial t} \mathbf{r}
\]

although this is not zero, it is a quadrupole term of the same order as the term we dropped when we truncated the expansion to linear order.
\[ \mathbf{S} \quad \int d^3r \mathbf{\hat{r}} \times \mathbf{r} = - \int d^3r \mathbf{\hat{r}} \times \mathbf{r} \quad \text{ignoring the quadrupole term} \]

\[ = \frac{1}{2} \int d^3r \left[ \mathbf{\hat{r}} \cdot \mathbf{r} - \mathbf{r} \cdot \mathbf{\hat{r}} \right] \]

\[ \sum_{\text{cen}} g_i \mathbf{r}_{ni} \cdot \mathbf{\hat{r}}_{ni} = \frac{1}{2} \sum_{\text{cen}} g_i \left[ \mathbf{r}_{ni} \cdot \mathbf{\hat{r}}_{ni} - \mathbf{\hat{r}}_{ni} \cdot \mathbf{r}_{ni} \right] \]

\[ = - \nabla f(r - \mathbf{r}_n) \cdot \sum_{\text{cen}} g_i \mathbf{r}_{ni} \cdot \mathbf{\hat{r}}_{ni} = - \nabla f(r - \mathbf{r}_n) = \frac{1}{2} \sum_{\text{cen}} g_i \left[ \mathbf{r}_{ni} \cdot \mathbf{\hat{r}}_{ni} - \mathbf{\hat{r}}_{ni} \cdot \mathbf{r}_{ni} \right] \]

\[ = - \frac{1}{2} \sum_{\text{cen}} g_i \left[ (\nabla f \cdot \mathbf{r}_{ni}) \mathbf{\hat{r}}_{ni} - (\nabla f \cdot \mathbf{\hat{r}}_{ni}) \mathbf{r}_{ni} \right] \]

\[ = - \frac{1}{2} \sum_{\text{cen}} g_i \nabla f \times (\mathbf{\hat{r}}_{ni} \times \mathbf{r}_{ni}) \quad \text{triple product rule} \]

\[ = \nabla f(r - \mathbf{r}_n) \times \frac{1}{2} \sum_{\text{cen}} \mathbf{r}_{ni} \times \mathbf{\hat{r}}_{ni} g_i \]

\[ = \nabla f(r - \mathbf{r}_n) \times \frac{1}{2} \int d^3r \mathbf{r} \times \mathbf{\hat{r}} \]

\[ = \nabla f(r - \mathbf{r}_n) \times c \mathbf{m}_n \quad \text{where} \quad \mathbf{m}_n = \frac{1}{2c} \sum_{\text{cen}} \mathbf{r}_{ni} \times \mathbf{\hat{r}}_{ni} g_i \]

is magnetic dipole moment of molecule \( n \)

\[ = \nabla \times f(r - \mathbf{r}_n) c \mathbf{m}_n \]

\[ = \nabla \times \left( c \mathbf{m}_n \delta (r - \mathbf{r}_n) \right) \]
Adding all the processes

\[ \langle \mathbf{f}_n \rangle = \left( \mathbf{g}_n \mathbf{v}_n \right) \delta (r - R_n) + \mathbf{c} \nabla \times \left( \mathbf{m}_n \delta (r - R_n) \right) \]

\[ + \frac{2}{\alpha} \begin{array}{c} \langle \mathbf{p}_n \delta (r - R_n) \rangle \\ \langle (\mathbf{v}_n \cdot \mathbf{v}) \rangle \end{array} \langle \mathbf{p}_n \delta (r - R_n) \rangle \\
- \mathbf{v}_n \nabla \cdot \left( \mathbf{p}_n \delta (r - R_n) \right) \]

Define \( \mathbf{M}(r) = \sum_n \langle \mathbf{m}_n \delta (r - R_n) \rangle \) average magnetization density

\( \mathbf{P}(r) = \sum_n \langle \mathbf{p}_n \delta (r - R_n) \rangle \) polarization density, as before

\[ \sum_n \langle \mathbf{f}_n \rangle = \sum_n \left( \mathbf{g}_n \mathbf{v}_n \right) \delta (r - R_n) + \mathbf{c} \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \]

\[ + \sum_n \left( (\mathbf{v}_n \cdot \mathbf{v}) \langle \mathbf{p}_n \delta (r - R_n) \rangle - \mathbf{v}_n \nabla \cdot \left( \mathbf{p}_n \delta (r - R_n) \right) \right) \]

See Jackson (6.96) for additional electric quadrupole terms

The last term on the right hand side is usually small and ignored. This is because the molecular velocities \( \mathbf{v}_n \) are usually small, and randomly oriented, so that they average to zero.

(See Jackson (6.100) for case of net translation of dielectrics: \( \mathbf{v}_n = \text{const, \ all \ } n \) )
Define macroscopic current density

\[ j(\vec{r},t) = \sum_{i=1}^{\text{free}} \nabla \cdot \vec{v}_{i}(\vec{r}-\vec{r}_i) + \sum_{\text{molecules}}^{\text{up}} n_i \vec{v}_i \cdot \delta(\vec{r}-\vec{r}_i) \]

Current of free charges current of molecular drift

If molecules are charged

Then \[ \langle j_0 \rangle = \vec{j} + c \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \]

Amperes's law becomes upon averaging

\[ \nabla \times \vec{B} = \frac{4\pi}{c} \langle j_0 \rangle + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

\[ = \frac{4\pi}{c} j + 4\pi \nabla \times \vec{M} + \frac{4\pi}{c} \frac{\partial \vec{P}}{\partial t} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

\[ \nabla \times (\vec{B} - 4\pi \vec{M}) = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial}{\partial t} \left( \vec{E} + 4\pi \vec{P} \right) \]

Define \[ \vec{H} = \vec{B} - 4\pi \vec{M} \]

As before

\[ \nabla \times \vec{H} = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

Official nomenclature: \( \vec{B} \) is the magnetic induction

\( \vec{H} \) is the magnetic field

Common usage: both \( \vec{H} \) and \( \vec{B} \) are called magnetic field