When atoms have intrinsic magnetic moments due to electron spin, we can add these to $\vec{M}$ in obvious way

When molecules are neutral, $q_n = 0$, the "bound current" is given by

$$\vec{j}_{bound} = \sum_n \langle \vec{j}_n \rangle = c\,\vec{\nabla}\times\vec{M} + \frac{\partial \vec{P}}{\partial t}$$

Note that the $\frac{\partial \vec{P}}{\partial t}$ term is crucial to give conservation of bound charge

$$\vec{\nabla}\cdot\vec{j}_{bound} = c\,\vec{\nabla}\cdot(\vec{\nabla}\times\vec{M}) + \vec{\nabla}\cdot\frac{\partial \vec{P}}{\partial t}$$

$$= 0 + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{P})$$

$$= -\frac{\partial \rho_{bound}}{\partial t}$$ where $\rho_{bound} = -\vec{\nabla}\cdot\vec{P}$ is bound charge density

So $$\boxed{\vec{\nabla}\cdot\vec{j}_{bound} + \frac{\partial \rho_{bound}}{\partial t} = 0}$$

and bound charge is conserved.

Since total average charge must be conserved, ie

$$\vec{\nabla}\cdot\langle \vec{j}_0 \rangle - \frac{\partial \langle \rho_0 \rangle}{\partial t} = 0,$$ and $\langle \vec{j}_0 \rangle = \vec{j} + \vec{j}_{bound}$

free current

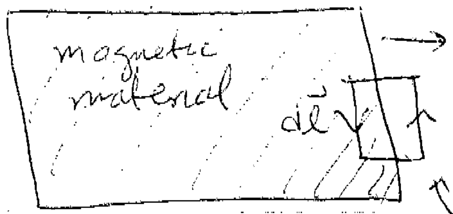
$\langle \rho_0 \rangle = \rho + \rho_{bound}$

free charge

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{j} + \frac{\partial \rho}{\partial t} = 0}$$

free charge is also conserved

At a surface of a magnetic material



$\longrightarrow \hat{n}$ outward normal to surface

take $\hat{z} \equiv \hat{dl} \times \hat{n}$ out of page

Amperean loop C bounding surface S of area da

$$c\int_S da\, \hat{z}\cdot(\vec{\nabla}\times\vec{M}) = \int_S da\, \hat{z}\cdot\vec{J}_{bound} = da\, \hat{z}\cdot\vec{J}_{bound}$$

$$= (d\vec{l}\times\hat{n})\cdot\vec{K}_{bound} \quad \text{as width of loop} \to 0$$

$$= (\hat{n}\times\vec{K}_{bound})\cdot d\vec{l}$$

But by Stokes theorem

$$c\int_S da\, \hat{z}\cdot(\vec{\nabla}\times\vec{M}) = c\int_C d\vec{l}\cdot\vec{M} = c\, d\vec{l}\cdot\vec{M}$$ since width $\to 0$ and $\vec{M}=0$ outside

$$\Rightarrow \quad c\, d\vec{l}\cdot\vec{M} = (\hat{n}\times\vec{K}_{bound})\cdot d\vec{l}$$ for any $d\vec{l}$ in plane of surface

$$\Rightarrow \quad c\vec{M}_t = \hat{n}\times\vec{K}_{bound}$$

where $\vec{M}_t$ is component of $\vec{M}$ tangential to the surface (since $\vec{K}_b$ is in plane of surface, $\hat{n}\times\vec{K}$ is also entirely in the plane of the surface)

$$\Rightarrow \quad c\,\hat{n}\times\vec{M}_t = c\,\hat{n}\times\vec{M} = \hat{n}\times(\hat{n}\times\vec{K}_{bound}) = -\vec{K}_{bound}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \vec{K}_{bound} &= c\,\vec{M}\times\hat{n} \\ \vec{J}_{bound} &= c\,\vec{\nabla}\times\vec{M} \end{aligned}}$$

Total bound charge vanishes

$$Q_{bound} = \int_V d^3r\ \rho_{bound} + \int_S da\ \sigma_{bound}$$

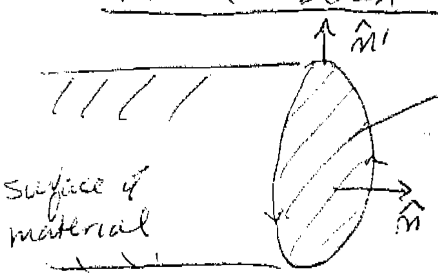
$\uparrow$ vol of dielectric $\quad\leftarrow$ surface of dielectric

$$= \int_V d^3r\ -\vec{\nabla}\cdot\vec{P} + \int da\ \hat{n}\cdot\vec{P}$$

but by Gauss theorem $\int_V d^3r\ \vec{\nabla}\cdot\vec{P} = \int da\ \hat{n}\cdot\vec{P}$

so $Q_{bound} = -\int da\ \hat{n}\cdot\vec{P} + \int da\ \hat{n}\cdot\vec{P} = 0$

Total bound current vanishes



crossectional area of magnetic material

S - bounding curve is C

$\hat{n}$ is normal to crossecti[on]
$\hat{n}'$ is normal to surfa[ce]

total current flowing through S is

$$\int_S da\ \hat{n}\cdot\vec{j}_{bound} + \int_C dl\ \vec{K}_{bound}\cdot\hat{n}$$

$$= c\int_S da\ \hat{n}\cdot(\vec{\nabla}\times\vec{M}) + c\int_C dl\ \hat{n}\cdot(\vec{M}\times\hat{n}')$$

$$= c\int_C d\vec{l}\cdot\vec{M} + c\int_C dl\ \underbrace{(\hat{n}'\times\hat{n})}_{=-\hat{t}}\cdot\vec{M}$$

$-\hat{t}$ unit tangent, $d\vec{l} = dl\ \hat{t}$

$$= c\int_C d\vec{l}\cdot\vec{M} - c\int_C d\vec{l}\cdot\vec{M} = 0$$

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

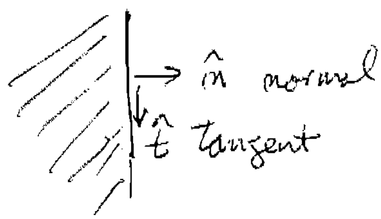
$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t} \qquad \vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

where ρ and $\vec{j}$ are macroscopic charge + current densities
do not include bound charges or currents

$\vec{D} = \vec{E} + 4\pi\vec{P}$, $\vec{P}$ is polarization density
$\vec{H} = \vec{B} - 4\pi\vec{M}$, $\vec{M}$ is magnetization density

Boundary conditions for statics

electrostatics : at surface of a dielectric, or at interface between two different dielectrics



$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \hat{t}\cdot\vec{E}_{above} = \hat{t}\cdot\vec{E}_{below}$
tangential component $\vec{E}$ is continuous
$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \Rightarrow \hat{n}\cdot(\vec{D}_{above} - \vec{D}_{below}) = 4\pi\sigma$
normal component of $\vec{D}$ jumps by $4\pi\sigma$

magnetostatics : at surface or interface of magnetic materials

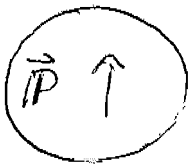
$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \hat{n}\cdot\vec{B}_{above} = \hat{n}\cdot\vec{B}_{below}$
normal component of $\vec{B}$ is continuous
$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} \Rightarrow \hat{t}\cdot(\vec{H}_{above} - \vec{H}_{below}) = \frac{4\pi}{c}\vec{K}$
tangential component of $\vec{H}$ jumps by $\frac{4\pi}{c}\vec{K}$

if $\sigma = 0$, ie no free surface charge, then $\hat{n}\cdot\vec{D}$ continuous

if $\vec{K} = 0$, ie no free surface current, then $\hat{t}\cdot\vec{H}$ continuous

Examples

① Uniformly polarized sphere of radius R $\qquad \vec{\mathbb{P}} = \mathbb{P}\,\hat{z}$



bound charge $\quad \rho_b = -\vec{\nabla}\cdot\vec{\mathbb{P}} = 0 \quad$ as $\vec{\mathbb{P}}$ constant

$$\sigma_b = \hat{n}\cdot\vec{\mathbb{P}} = \hat{r}\cdot\vec{\mathbb{P}} = \mathbb{P}\cos\theta$$

we saw earlier that a sphere with surface charge $\sigma(\theta) = \sigma_0 \cos\theta$ gives an electric field like a pure dipole for $r \geq R$, and is constant for $r < R$.

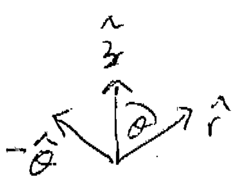
$$\vec{E}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 \mathbb{P}\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r > R \\ -\dfrac{4\pi\mathbb{P}}{3}\,\hat{z} & r < R \end{cases}$$

total dipole moment is $\vec{p} = \frac{4}{3}\pi R^3 \vec{\mathbb{P}}$

check behavior at boundary

Tangential component $\vec{E}$

$$\vec{E}^{\,t}_{above} = \left(\tfrac{4}{3}\pi R^3 \mathbb{P}\right)\frac{\sin\theta\,\hat{\theta}}{R^3} = \frac{4\pi\mathbb{P}}{3}\sin\theta\,\hat{\theta}$$



$$\vec{E}^{\,t}_{below} = -\frac{4\pi\mathbb{P}}{3}(\hat{z}\cdot\hat{\theta})\,\hat{\theta} = \frac{4}{3}\pi\mathbb{P}\sin\theta\,\hat{\theta}$$

$\Rightarrow$ Tangential component $\vec{E}$ is continuous

normal component of $\vec{D}$

outside: $\vec{\mathbb{P}} = 0 \;\Rightarrow\; \vec{D} = \vec{E}$

$$\Rightarrow \hat{n}\cdot\vec{D} = \hat{r}\cdot\vec{E} = \left(\tfrac{4}{3}\pi R^3 \mathbb{P}\right)\frac{2\cos\theta\,\hat{r}}{R^3} = \frac{8}{3}\pi\,\mathbb{P}\cos\theta,$$

inside: $\vec{E} = -\dfrac{4\pi\vec{\mathbb{P}}}{3} \Rightarrow \vec{\mathbb{P}} = -\dfrac{3}{4\pi}\vec{E}$

$$\vec{D} = \vec{E} + 4\pi\vec{\mathbb{P}} = \vec{E} - 3\vec{E} = -2\vec{E} = \frac{8\pi}{3}\mathbb{P}\,\hat{z}$$

$$\hat{m}\cdot\vec{D} = \hat{r}\cdot\left(\tfrac{8\pi}{3}\mathbb{P}\,\hat{z}\right) = \tfrac{8\pi}{3}\mathbb{P}\cos\theta$$

$\Rightarrow$ normal component $\vec{D}$ is continuous

Note: normal component of $\vec{E}$ should jump by $4\pi\sigma_b = 4\pi\mathbb{P}\cos\theta$

to check this: $\hat{m}\cdot\vec{E}^{\,\text{inside}} = \hat{r}\cdot\left(-\tfrac{4}{3}\pi\mathbb{P}\,\hat{z}\right) = -\tfrac{4}{3}\pi\mathbb{P}\cos\theta$

$$\hat{m}\cdot\left(\vec{E}^{\text{above}} - \vec{E}^{\text{below}}\right) = \tfrac{8}{3}\pi\mathbb{P}\cos\theta + \tfrac{4}{3}\pi\mathbb{P}\cos\theta$$

$$= \frac{12}{3}\pi\mathbb{P}\cos\theta = 4\pi\mathbb{P}\cos\theta = 4\pi\sigma_b(\theta)$$

(2) Uniformly magnetized sphere of radius R $\quad \vec{M} = M\hat{z}$

bound current $\quad \vec{J}_b = c\,\vec{\nabla}\times\vec{M} = 0$ as $\vec{M}$ constant

$$\vec{K}_b = c\,\vec{M}\times\hat{n} = cM\,(\hat{z}\times\hat{r})$$
$$= cM\sin\theta\,\hat{\varphi}$$

we saw earlier that a sphere with surface current $\vec{K}_b = K_o \sin\theta\,\hat{\theta}$ gives a magnetic field that is pure dipole for $r>R$, and is constant for $r<R$.

$$\vec{B}(\vec{r}) = \begin{cases} \left(\frac{4}{3}\pi R^3 M\right)\left[\dfrac{2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}}{r^3}\right] & r>R \\ \frac{8}{3}\pi M\,\hat{z} & r<R \end{cases}$$

total dipole moment is $\vec{m} = \frac{4}{3}\pi R^3 \vec{M}$

Check behavior at boundary

normal component of $\vec{B}$

$$\hat{m} \cdot \vec{B}^{\text{above}} = \hat{r} \cdot \vec{B}^{\text{above}} = \frac{8}{3}\pi M \cos\theta$$

$$\hat{m} \cdot \vec{B}^{\text{below}} = \hat{r} \cdot \vec{B}^{\text{below}} = \frac{8}{3}\pi M (\hat{r} \cdot \hat{z}) = \frac{8}{3}\pi M \cos\theta$$

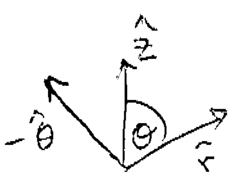
$\Rightarrow$ normal component of $\vec{B}$ is continuous

Tangential component of $\vec{H}$

outside: $\vec{M} = 0 \Rightarrow \vec{H} = \vec{B}$

$$\vec{H}^{t}_{\text{above}} = \left(\frac{4}{3}\pi M\right) \sin\theta \, \hat{\theta}$$

inside:
$$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi\left(\frac{3}{8\pi}\vec{B}\right) = \vec{B} - \frac{3}{2}\vec{B} = -\frac{1}{2}\vec{B}$$
$$= -\frac{4\pi}{3} M \hat{z}$$



so
$$\vec{H}^{t}_{\text{below}} = -\frac{4\pi}{3} M (\hat{z} \cdot \hat{\theta}) = \frac{4\pi}{3} M \sin\theta \, \hat{\theta}$$

$\Rightarrow$ Tangential component $\vec{H}$ is continuous

Note: tangential component $\vec{B}$ should jump by $\frac{4\pi}{c}\vec{K}_b = 4\pi M \sin\theta \, \hat{\theta}$

inside:

to check:
$$\vec{B}^{t}_{\text{below}} = \frac{8}{3}\pi M (\hat{z} \cdot \hat{\theta})\hat{\theta} = -\frac{8}{3}\pi M \sin\theta \, \hat{\theta}$$

$\vec{H}^{t}_{\text{above}} = \vec{B}^{t}_{\text{above}} \Rightarrow$
$$\vec{B}^{t}_{\text{above}} - \vec{B}^{t}_{\text{below}} = \frac{4}{3}\pi M \sin\theta \, \hat{\theta} + \frac{8}{3}\pi M \sin\theta \, \hat{\theta}$$
$$= 4\pi M \sin\theta \, \hat{\theta} = \frac{4\pi}{c}\vec{K}_b$$

Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

where ρ and $\vec{j}$ are macroscopic charge + current densities and

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \vec{P} \text{ is polarization density}$$
$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad \vec{M} \text{ is magnetization density}$$

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In some materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

<u>Ferromagnet</u>: $\vec{M}$ can be non zero even if $\vec{B} = 0$

<u>Ferroelectric</u>: $\vec{P}$ can be non zero even if $\vec{E} = 0$

But more common are <u>linear</u> materials in which, for <u>small</u> $\vec{E}$ and $\vec{B}$, one has $\vec{P} \propto \vec{E}$ and $\vec{M} \propto \vec{B}$.

linear dielectric

$\vec{P} = \chi_e \vec{E}$ χ_e is "electric susceptibility"

$\chi_e > 0$ for statics

$\vec{D} = \vec{E} + 4\pi \vec{P} = (1 + 4\pi \chi_e)\vec{E}$

$\vec{D} = \epsilon \vec{E}$ with $\epsilon = 1 + 4\pi \chi_e$

ϵ is the dielectric constant

linear magnetic materials

$\vec{M} = \chi_m \vec{H}$ χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$\vec{H} = \vec{B} - 4\pi \vec{M} = \vec{B} - 4\pi \chi_m \vec{H}$

$\vec{B} = (1 + 4\pi \chi_m)\vec{H}$

$\vec{B} = \mu \vec{H}$ with $\mu = 1 + 4\pi \chi_m$

μ is magnetic permeability

For statics, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are constants depending on the material.

When we consider dynamics we will see that ϵ becomes a function of frequency.