Conservation of Energy — leave macroscopic Maxwell eqns for present. $\vec{E}, \vec{B}, \rho, \vec{j}$ are now the exact microscopic quantities

Consider a collection of charged particles, described by charge density ρ and current density $\vec{j}$. The particles are contained in a volume V.

Define $\mathcal{E}_{mech}$ as total "mechanical" energy of the particles. $\mathcal{E}_{mech}$ = sum of particles kinetic energy plus potential energy of any non electromagnetic forces.

The particles will exert forces on each other via their electromagnetic interactions, ie via the $\vec{E}$ and $\vec{B}$ fields that they create. Define W as the work done on the particles by all electromagnetic forces. Then, by the work energy theorem of mechanics:

$$\frac{d\mathcal{E}_{mech}}{dt} = \frac{dW}{dt}$$

For a single charge q_i, (at $\vec{r}_i$ with velocity $\vec{v}_i$) $\frac{dW}{dt} = \vec{F}_i \cdot \vec{v}_i$

$$= q_i \vec{E}(\vec{r}_i) \cdot \vec{v}_i + q\underbrace{\left(\frac{\vec{v}_i}{c} \times \vec{B}\right) \cdot \vec{v}_i}_{=0}$$

$$= q_i \vec{E}(r_i) \cdot \vec{v}_i$$

For the collection of charges, with

$$\vec{j}(\vec{r},t) = \sum_i q_i \vec{v}_i \, \delta(\vec{r}-\vec{r}_i(t))$$

the total rate of work done is

$$\frac{dW}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}(\vec{r}_i) = \int_V d^3r \, \vec{j} \cdot \vec{E}$$

So

$$\frac{d\mathcal{E}_{mech}}{dt} = \int_V d^3r \, \vec{j} \cdot \vec{E}$$

By maxwell equation $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$
we can write

$$\vec{j} = \frac{c}{4\pi}\left[\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right]$$

$$\int_V d^3r \, \vec{j}\cdot\vec{E} = \int_V d^3r \, \frac{c}{4\pi}\left[(\vec{\nabla}\times\vec{B})\cdot E - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\cdot\vec{E}\right]$$

use $\frac{\partial \vec{E}}{\partial t}\cdot E = \frac{1}{2}\frac{\partial E^2}{\partial t}$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot B - E\cdot(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = (\vec{\nabla}\times\vec{E})\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

then use $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$$\text{So } \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\cdot\vec{B} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

$$= -\frac{1}{2c}\frac{\partial B^2}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$$

Combine results to get

$$\int_V d^3r \; \vec{j}\cdot\vec{E} = -\frac{1}{4\pi}\int_V d^3r \left[\frac{1}{2}\frac{\partial B^2}{\partial t} + \frac{1}{2}\frac{\partial E^2}{\partial t} + c\,\vec{\nabla}\cdot(\vec{E}\times\vec{B}) \right]$$

define $\boxed{\begin{array}{l} u = \frac{1}{8\pi}(E^2+B^2) \\ \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} \end{array}}$ electromagnetic energy density / Poynting vector - energy current

then

$$\frac{d\mathcal{E}_{mech}}{dt} = \int_V d^3r\,\vec{j}\cdot\vec{E} = -\int_V d^3r \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} \right]$$

If we define $\mathcal{E}_{EM}$, the electromagnetic energy of the volume V, as

$$\mathcal{E}_{EM} = \int_V d^3r \; u$$

then

$$\frac{d}{dt}\left(\mathcal{E}_{mech} + \mathcal{E}_{EM}\right) = -\oint_S da\;\hat{n}\cdot\vec{S}$$

~~or if we write $\frac{\partial u_{mech}}{\partial t} = \vec{j}\cdot\vec{E}$ as the rate of change of mechanical~~

or we can write in differential form

$$\vec{j}\cdot\vec{E} + \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0$$

local energy conservation law if interpret $\vec{S}$ as energy current and u as EM energy density

rate of change of mechanical energy per unit volume

$$\frac{d}{dt}\left(\mathcal{E}_{mech} + \mathcal{E}_{EM}\right) = -\oint_S da\ \hat{n}\cdot\vec{S}$$

total energy in V can decrease only if electromagnetic energy is being transported through the surface S by the EM energy current $\vec{S}$.
assumes the charged particles do not leave the volume V.

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under certain conditions, we can derive a similar conservation law for the macroscopic maxwell eqns.

Consider that $\vec{j}$ is current of the free (charged) particles.

Then repeating the above steps:

$$\int_V d^3r\ \vec{j}\cdot\vec{E} = \frac{c}{4\pi}\int d^3r\ \vec{E}\cdot\left[\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right]$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H})$$
$$= -\frac{1}{c}H\cdot\frac{\partial\vec{B}}{\partial t} - E\cdot\vec{\nabla}\times\vec{H}$$

so

$$\int_V d^3r\ \vec{j}\cdot\vec{E} = -\frac{1}{4\pi}\int_V d^3r\left[c\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right.$$

If the medium is linear, and we have quasistatic conditions, so that

$$\vec{D}(t) \simeq \epsilon\vec{E}(t)$$
$$\vec{H}(t) \simeq \frac{1}{\mu}\vec{B}(t)$$

then we can write

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \epsilon\,\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D})$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{\mu}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2\mu}\frac{\partial B^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{H})$$

** Note: in general, as we will soon see, above conditions are not satisfied. ϵ will depend on frequency ω and $\vec{D}(t)$ and $\vec{E}(t)$ are non locally related in time.
$\vec{D}(t) = \int_{-\infty}^{t} dt'\,\epsilon(t-t')\,\vec{E}(t')$. Only at low frequencies,

ie in quasistatic case, can we write $\vec{D}(t) \simeq \epsilon(\omega=0)\,\vec{E}(t)$

Assuming the above conditions are met, then

$$\int_V d^3r\;\vec{j}\cdot\vec{E} + \int_V d^3r\,\frac{\partial u}{\partial t} = -\oint_S da\;\hat{n}\cdot\vec{S}$$

where
$$\begin{cases} u = \dfrac{1}{8\pi}\left[\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right] \\ \vec{S} = \dfrac{c}{4\pi}\left[\vec{E}\times\vec{H}\right] \end{cases}$$

$\Rightarrow$ electromagnetic energy in dielectric + magnetic materials under quasistatic conditions is

$$\int_V d^3r\left[\underbrace{\frac{1}{8\pi}\vec{E}\cdot\vec{D}}_{\text{electrostatic energy}} + \underbrace{\frac{1}{8\pi}\vec{B}\cdot\vec{H}}_{\text{magnetostatic energy}}\right]$$