Statics

Electrostatic Energy

Returning to microscopic fields and charges

$$\mathcal{E} = \frac{1}{8\pi}\int_V d^3r\, E^2 \qquad \text{use } \vec{E} = -\vec{\nabla}\phi$$

$$= \frac{-1}{8\pi}\int_V d^3r\, (\vec{\nabla}\phi)\cdot\vec{E} \qquad \text{use } \vec{\nabla}\cdot(\phi\vec{E}) = \phi\,\vec{\nabla}\cdot\vec{E} + (\vec{\nabla}\phi)\cdot\vec{E}$$

$$= \frac{-1}{8\pi}\int_V d^3r\left[\vec{\nabla}\cdot(\phi\vec{E}) - \phi\,\vec{\nabla}\cdot\vec{E}\right] \qquad \text{use } \vec{\nabla}\cdot E = 4\pi\rho$$

$$= \frac{1}{2}\int_V d^3r\, \rho\phi - \frac{1}{8\pi}\oint_S da\, \hat{n}\cdot\phi\vec{E} \qquad \text{by Gauss Theorem}$$

If let V be all space, $S\to\infty$, then $\phi\sim\frac{1}{r}$, $E\sim\frac{1}{r^2}$

surface integral $\sim \frac{R^2}{R^3}\to 0$ as $R\to\infty$.

$$\boxed{\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\phi}$$

can also use $\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ to write

$$\boxed{\mathcal{E} = \frac{1}{2}\int d^3r\, d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

charge-charge interaction

Magnetostatic Energy

microscopic fields and currents

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\ B^2 \qquad \text{use } \vec{B} = \vec{\nabla}\times\vec{A}$$

$$= \frac{1}{8\pi}\int d^3r\ \vec{B}\cdot\vec{\nabla}\times\vec{A} \qquad \text{use } \vec{\nabla}\cdot(\vec{B}\times\vec{A}) = \vec{A}\cdot(\vec{\nabla}\times\vec{B}) - \vec{B}\cdot(\vec{\nabla}\times\vec{A})$$

$$= \frac{1}{8\pi}\int d^3r \left[\vec{A}\cdot\vec{\nabla}\times\vec{B} - \vec{\nabla}\cdot(\vec{B}\times\vec{A})\right] \qquad \text{use } \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

$$= \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A} - \frac{1}{8\pi}\oint_S da\ \hat{n}\cdot(\vec{B}\times\vec{A})$$

as take V to fill all space, $S\to\infty$, surface term vanishes

$$\boxed{\mathcal{E} = \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A}}$$

In Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$, $\vec{A}(\vec{r}) = \int d^3r'\ \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

In any other gauge we have $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ for some scalar χ. So we can always write

$$\vec{A}(\vec{r}) = \int d^3r'\ \frac{\vec{j}(\vec{r}')}{c\,|\vec{r}-\vec{r}'|} + \vec{\nabla}\chi$$

regardless of the choice of gauge, where χ is then determined so $\vec{A}$ satisfies the desired gauge condition

$$\mathcal{E} = \frac{1}{2c}\int d^3r\, d^3r' \, \frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}\,')}{c\,|\vec{r}-\vec{r}\,'|} + \frac{1}{2c^2}\int d^3r\, \vec{j}\cdot\vec{\nabla}\chi$$

2nd term is $\int d^3r\, \vec{j}\cdot\nabla\chi = \int d^3r \left[\nabla\cdot(\vec{j}\chi) - \chi\,\vec{\nabla}\cdot\vec{j}\right]$

$$= \underbrace{\oint_S da\, \hat{n}\cdot\vec{j}\chi}_{\text{vanishes as } S\to\infty} - \underbrace{\int d^3r\, \chi\,\vec{\nabla}\cdot\vec{j}}_{\text{vanishes in magnetostatics where } \vec{\nabla}\cdot\vec{j}=0}$$

So

$$\boxed{\mathcal{E} = \frac{1}{2c^2}\int d^3r\, d^3r' \, \frac{\vec{j}(\vec{r})\cdot\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}}$$

current-current interaction

Momentum Conservation

For charges q_i at positions $\vec{r}_i$ with velocities $\dot{\vec{r}}_i$

$$\frac{d\vec{P}^{mech}}{dt} = \sum_i \vec{F}_i = \sum_i q_i\left(\vec{E}(\vec{r}_i) + \frac{1}{c}\vec{v}_i \times \vec{B}(\vec{r}_i)\right)$$

"mechanical" momentum of the charges

force on charge i

$$= \int_V d^3r \left[\rho\vec{E} + \frac{1}{c}\vec{j}\times\vec{B}\right]$$

$$\rho E + \frac{1}{c}\vec{j}\times\vec{B} = \frac{1}{4\pi}\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \left(\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)\times\vec{B}\right]$$

Now $\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) + \frac{1}{c}\left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$ use $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

$$= \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) - \vec{E}\times(\vec{\nabla}\times\vec{E})$$

So $-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\times\vec{B} = -\vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$

Therefore

$$\rho E + \frac{1}{c}\vec{j}\times\vec{B} = \frac{1}{4\pi}\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\overbrace{(\vec{\nabla}\cdot\vec{B})}^{=0} - \vec{B}\times(\vec{\nabla}\times\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})\right]$$

Define electromagnetic momentum density

$$\boxed{\vec{\Pi} = \frac{1}{4\pi c}\vec{E}\times\vec{B} = \frac{1}{c^2}\vec{S}}$$

($\vec{S}$ is Poynting vector)

then

$$\frac{d\vec{P}^{mech}}{dt} + \frac{d}{dt}\int_V d^3r\,\vec{\Pi} = \frac{1}{4\pi}\int_V d^3r\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + \vec{B}(\vec{\nabla}\cdot\vec{B}) - \vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

want to rewrite as a surface integral

ith component of integrand on right hand side is ($\vec{E}$ part only)
(sum over repeated indicies)

$$E_i \partial_j E_j - \varepsilon_{ijk} E_j \varepsilon_{klm} \partial_l E_m$$

$$= E_i \partial_j E_j - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) E_j \partial_l E_m$$

$$= E_i \partial_j E_j - E_j \partial_i E_j + E_j \partial_j E_i$$

$$= \partial_j \left(E_i E_j - \tfrac{1}{2}\delta_{ij} E^2 \right)$$

Define <u>Maxwell's stress tensor</u>

$$\boxed{T_{ij} = \frac{1}{4\pi}\left[E_i E_j + B_i B_j - \tfrac{1}{2}\delta_{ij}(E^2 + B^2) \right]}$$

(note $T_{ij} = T_{ji}$
Symmetric tensor)

Then

$$\frac{d}{dt} p_i^{mech} + \frac{d}{dt}\int_V d^3r\, \Pi_i = \int_V d^3r\, \partial_j T_{ij} \qquad \left(\partial_j T_{ij} = \frac{\partial T_{ij}}{\partial x_j}\right)$$

$$= \oint_S da\, T_{ij}\, \hat{n}_j$$

$$\frac{d}{dt}\vec{p}^{\,mech} + \frac{d}{dt}\int_V d^3r\, \vec{\Pi} = \oint_S da\, \overleftrightarrow{T}\cdot\hat{n}$$

- T_{ij} gives the flow of the i^{th} component of electromagnetic field momentum through an element of surface area $\perp$ ~~to~~ to direction $\hat{e}_j$

~~For static situations where~~ ~~$\frac{\partial\vec{\Pi}}{\partial t} = 0$, $\frac{d\vec{P}^{mech}}{dt} = \vec{F}_{tot} = \oint_S da\, \overleftrightarrow{T}\cdot\hat{n}$~~
~~gives electromagnetic force on the surface S~~

Note: $\frac{d\vec{P}^{mech}}{dt}$ is ~~also~~ equal to the total electromagnetic force on the volume V.

Hence we can write

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n} - \frac{d}{dt} \int_V d^3r\, \vec{\Pi}$$

for static situations, the 2nd term vanishes and

$$\vec{F}_{EM} = \oint_S da\, \overleftrightarrow{T} \cdot \hat{n}$$

$\overleftrightarrow{T}_{ij}$ is ith component of static force on unit area with normal $\hat{e}_j$

this is origin of the term "stress" tensor.
$\overleftrightarrow{T}$ is like the stress tensor of an elastic medium
T_{xx}, T_{yy}, T_{zz} are like pressure
off diagonal elements are like shear stresses

Force on a conductor surface

$\leftarrow \sigma$ surface charge on conductor

$\rightarrow \hat{n}$ net force on surface per unit area is

$$\vec{f} = \overleftrightarrow{T}^{above} \cdot \hat{n} - \overleftrightarrow{T}^{below} \cdot \hat{n}$$

$\uparrow = 0$ as $\vec{E} = 0$ inside conductor

$$\vec{f} = \frac{1}{4\pi}\left[\vec{E}(\vec{E}\cdot\hat{n}) - \frac{1}{2}\hat{n}E^2\right]$$

for conducting surface

$$\hat{n}\cdot\vec{E}^{above} = 4\pi\sigma \quad (\text{since } \vec{E} \text{ below} = 0)$$

and tangential component $\vec{E} = 0$

$$\Rightarrow \vec{E} = 4\pi\sigma\,\hat{n}$$

So $$\vec{f} = \frac{1}{4\pi}\left[(4\pi\sigma\hat{n})(4\pi\sigma) - \frac{1}{2}\hat{n}(4\pi\sigma)^2\right]$$

$$\cancel{\vec{f} = 2\pi\sigma^2\hat{n}}$$

$$\vec{f} = \frac{\hat{n}}{4\pi}\left[(4\pi\sigma)^2 - \frac{1}{2}(4\pi\sigma)^2\right] = 2\pi\sigma^2\hat{n}$$

force per unit area $$\boxed{\vec{f} = 2\pi\sigma^2\hat{n} = \frac{1}{2}\sigma\vec{E}}$$

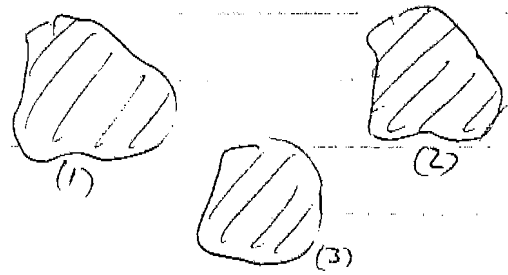
$\vec{f} = \sigma\vec{E}_{ave}$

where $\vec{E}_{ave} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below})$ is average field at surface averaging over above + below

Note factor ½. Naively one might have thought $\vec{f} = \sigma\vec{E}$. But need to exclude self field of charge on surface from acting on itself. See also Jackson pg 42 for another approach

Capacitance

Consider a set of conductors with potential $\phi(\vec{r}) = V_i$ fixed on conductor i



(also need condition on $\phi(\vec{r}) \to \infty$ if system is not enclosed)

From uniqueness theorem we know that specifying the V_i on each conductor is enough to determine the potential $\phi(\vec{r})$ everywhere. We can write this potential in the following form -

Let $\phi^{(i)}(\vec{r})$ be the solution to the boundary value problem
$\nabla^2\phi^{(i)}(r) = 0$ and $\phi^{(i)}(\vec{r}) = \begin{cases} 1 & \text{if } \vec{r} \text{ on surface of conductor } (i) \\ 0 & \text{if } \vec{r} \text{ on surface of any other conductor } (j),\ j \neq i \end{cases}$

Then by superposition

$$\phi(\vec{r}) = \sum_i V_i\, \phi^{(i)}(\vec{r})$$

is solution to the problem $\nabla^2\phi = 0$ and $\phi(\vec{r}) = V_i$ for $\vec{r}$ on surface of conductor (i)

The surface charge density at $\vec{r}$ on surface of conductor (i) is

$$\sigma^{(i)}(\vec{r}) = \frac{-1}{4\pi}\frac{\partial \phi(\vec{r})}{\partial n} = -\frac{1}{4\pi}\sum_j V_j \frac{\partial \phi^{(j)}(\vec{r})}{\partial n}$$

where $\frac{\partial \phi}{\partial n} = (\vec{\nabla}\phi)\cdot\hat{n}$ is the derivative normal to the surface at point $\vec{r}$.

The total charge on conductor (i) is

$$Q_i = \int_{S_i} da\, \sigma^{(i)}(\vec{r}) = -\frac{1}{4\pi} \sum_j V_j \int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$$

surface of conductor (i)

Define $C_{ij} \equiv -\frac{1}{4\pi} \int_{S_i} da \frac{\partial \phi^{(j)}}{\partial n}$ — the C_{ij} depend only on the geometry of the conductors

Then we have

$$\boxed{Q_i = \sum_j C_{ij} V_j}$$

C_{ij} is the capacitance matrix

The charge on conductor (i) is a linear function of the potentials V_j on the conductors (j)

Since we know that specifying the Q_i that is on each conductor will uniquely determine $\phi(\vec{r})$ and hence the potential V_i on each conductor, the capacitance matrix is invertable

$$V_i = \sum_j [C^{-1}]_{ij} Q_j$$

The electrostatic energy of the conductors is then

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\, \Phi = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j$$

Common to define capacitance of two conductors by

$$C = \frac{Q}{V_1 - V_2}$$

when conductor (1) has charge Q
conductor (2) has charge $-Q$
$V_1 - V_2$ is potential difference between the two conductors.
all other conductors fixed at $V_i = 0$

We can determine C in terms of the elements of the matrix C_{ij}

$$\left.\begin{aligned} Q &= C_{11}V_1 + C_{12}V_2 \\ -Q &= C_{21}V_1 + C_{22}V_2 \end{aligned}\right\} \Rightarrow V_2 = -\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)V_1$$

$$\Rightarrow Q = \left[C_{11} - C_{12}\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right]V_1$$

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right]V_1$$

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12}\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)}{1 + \left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)}$$

$$\boxed{C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11} + C_{12} + C_{21} + C_{22}}}$$

Capacitance can also be defined when the space between the conductors is filled with a dielectric ϵ. In this case, if Q_i is the free charge, then Q_i/ϵ is the effective total charge to use in computing ϕ.

$$\Rightarrow \frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j$$ where $C_{ij}^{(0)}$ are capacitances appropriate to a vacuum between the conductors

$$\Rightarrow Q_i = \sum_j \epsilon C_{ij}^{(0)} V_j$$

$$= \sum_j C_{ij} V_j \quad \text{where } C_{ij} = \epsilon C_{ij}^{(0)}$$

the capacitance is increased by a factor the dielectric constant ϵ.