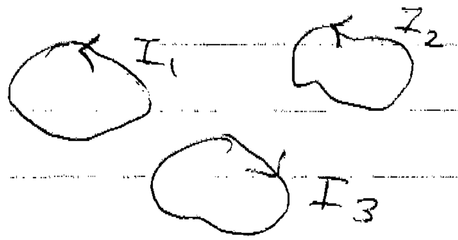
Inductance

Consider a set of current carrying loops C_i with currents I_i



In Coulomb gauge, we can write the magnetic vector potential $\vec{A}$ from these current loops as

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \, \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \sum_i \frac{I_i}{c} \oint_{C_i} \frac{d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

↑ integrate over loop C_i
integration variable is $\vec{r}'$

The magnetic flux through loop i is

$$\Phi_i = \int_{S_i} da \, \hat{n} \cdot \vec{B} = \int_{S_i} da \, \hat{n} \cdot \vec{\nabla} \times \vec{A} = \oint_{C_i} d\vec{l} \cdot \vec{A}$$

↑ surface bounded by loop C_i

$$\Phi_i = \sum_j \frac{I_j}{c} \oint_{C_i} \oint_{C_j} \frac{d\vec{l} \cdot d\vec{l}'}{|\vec{r}-\vec{r}'|}$$

pure geometrical quantity

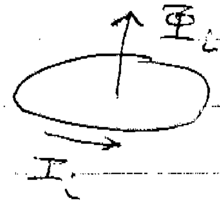
$$\boxed{\Phi_i \equiv c \sum_j M_{ij} I_j}$$

where $M_{ij} = \oint_i \oint_j \frac{d\vec{l}_i \cdot d\vec{l}_j}{c^2 |\vec{r}-\vec{r}'|}$

is the <u>mutual inductance</u> of loops (i) and (j). $M_{ij} = M_{ji}$

$L_i \equiv M_{ii}$ is self-inductance of loop (i)

The sign convention in the above is that, Φ_i is computed in direction given by right hand rule, according to the direction taken for current in loop (i)



Magnetostatic energy

$$\mathcal{E} = \frac{1}{2c}\int d^3r\ \vec{j}\cdot\vec{A} = \frac{1}{2c}\sum_i \oint_{C_i} d\vec{l}\cdot\vec{A}\ I_i$$

$$= \frac{1}{2c}\sum_i \Phi_i I_i$$

$$\mathcal{E} = \frac{1}{2}\sum_{i,j} M_{ij} I_i I_j$$

Force and torque on electric dipoles

localized charge distribution $\rho(\vec{r})$ with net charge $\int d^3r\, \rho = 0$

force on ρ in slowly varying electric field $\vec{E}$ is

$$\vec{F} = \int d^3r\, \rho(\vec{r})\, \vec{E}(\vec{r})$$

define $\vec{r} = \vec{r}_0 + \vec{r}\,'$ where $\vec{r}_0$ is some fixed reference point in center of charge distr ρ, and $\vec{r}\,'$ is distance relative to $\vec{r}_0$

$$\vec{F} = \int d^3r'\, \rho(\vec{r}\,')\, \vec{E}(\vec{r}_0 + \vec{r}\,')$$

since $\vec{E}$ is slowly varying on length scale where $\rho \neq 0$, we expand

$$\vec{F} \simeq \int d^3r'\, \rho(\vec{r}\,') \left[\vec{E}(\vec{r}_0) + (\vec{r}\,' \cdot \vec{\nabla}) \vec{E}(\vec{r}_0) \right] + \cdots$$

$$= \vec{E}(\vec{r}_0) \int d^3r'\, \rho(\vec{r}\,') + \left(\int d^3r'\, \rho(r')\, \vec{r}\,' \cdot \vec{\nabla} \right) \vec{E}(\vec{r}_0)$$

$$= 0 + (\vec{p} \cdot \vec{\nabla}) \vec{E}(\vec{r}_0)$$

$$\boxed{\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = \sum_{\alpha=1}^{3} p_\alpha \frac{\partial \vec{E}}{\partial r_\alpha}}$$

For $\vec{E} = \text{constant}$, $\vec{F} = 0$

Torque on ρ is

$$\vec{N} = \int d^3r\, \rho(\vec{r})\, \vec{r}\times\vec{E}(\vec{r}) \cong \int d^3r\, \rho(\vec{r})\, \vec{r}\times\left[\vec{E}(\vec{r}_0) + \cdots\right]$$

to lowest order $\boxed{\vec{N} = \vec{p}\times\vec{E}}$

Force and torque on magnetic dipoles

localized magnetostatic current distribution $\vec{j}(\vec{r})$

$$\vec{F} = \frac{1}{c}\int d^3r\, \vec{j}\times\vec{B}$$

expand about center of current $\vec{r}_0$

$$\vec{B}(\vec{r}) \simeq \vec{B}(\vec{r}_0) + (\vec{r}'\cdot\vec{\nabla})\vec{B}(\vec{r}_0) + \cdots$$

$$\vec{F} = \frac{1}{c}\left[\int d^3r'\, \vec{j}(\vec{r}')\right]\times\vec{B}(\vec{r}_0) + \frac{1}{c}\int d^3r'\, \vec{j}(\vec{r}')\times(\vec{r}'\cdot\vec{\nabla})\vec{B}(\vec{r}_0)$$

from discussion of magnetic dipole approx we had $\int d^3r\, \vec{j} = 0$ for magnetostatics where $\vec{\nabla}\cdot\vec{j} = 0$. So 1st term vanishes. The 2nd term can be written as

$$F_\alpha = \frac{\varepsilon_{\alpha\beta\gamma}}{c}\int d^3r'\, j_\beta\, r'_\delta\, \partial_\delta B_\gamma$$

for magnetostatics see magnetic dipole derivation

we need the tensor $\frac{1}{c}\int d^3r'\, j_\beta\, r'_\delta = -\frac{1}{c}\int d^3r'\, r'_\beta\, j_\delta$

$$= \frac{1}{2c}\int d^3r'\left[j_\beta r'_\delta - r'_\beta j_\delta\right]$$

$$= -m_\sigma\, \varepsilon_{\sigma\beta\delta}$$

↑ magnetic dipole $\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r}\times\vec{j}$

$$F_\alpha = \varepsilon_{\alpha\beta\gamma}\,\varepsilon_{\sigma\beta\delta}(-m_\sigma)\,\partial_\delta B_\gamma$$

$$= -\left(\delta_{\alpha\sigma}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\sigma\gamma}\right) m_\sigma \partial_\delta B_\gamma$$

$$= \underset{\alpha}{\vec\nabla}(\vec m\cdot\vec B) - \vec m_\alpha \vec\nabla\cdot\vec B$$

$$\boxed{\vec F = \vec\nabla(\vec m\cdot\vec B)} \qquad \text{as } \vec\nabla\cdot\vec B = 0$$

torque on $\vec j$ is

$$\vec N = \frac{1}{c}\int d^3r\ \vec r\times(\vec j\times\vec B)$$

to lowest order, $\vec B = \vec B(\vec r_0)$ is const over region where $\vec j \neq 0$

$$= \frac{1}{c}\int d^3r\left[\vec j\,(\vec r\cdot\vec B) - \vec B(\vec r\cdot\vec j)\right]$$

2^{nd} term $=0$ as follows

$$\int d^3r\ \vec r\cdot\vec j = \int d^3r\ \vec j\cdot\vec\nabla\left(\frac{r^2}{2}\right) \qquad \text{as } \vec\nabla\left(\frac{r^2}{2}\right) = \vec r$$

$$= -\int d^3r\,(\vec\nabla\cdot\vec j)\left(\frac{r^2}{2}\right)$$

integrate by parts. Surface term $\to 0$ as $\vec j$ is localized

$$= 0 \qquad \text{as } \vec\nabla\cdot\vec j = 0 \text{ in magnetostatics}$$

1^{st} term involves

see derivation of magnetic dipole approx

$$\int d^3r\ \vec j\,\vec r = -\int d^3r\ \vec r\,\vec j = \frac{1}{2}\int d^3r\left[\vec j\,\vec r - \vec r\,\vec j\right]$$

So

$$\vec N = \frac{1}{2c}\int d^3r\left[\vec j\,(\vec r\cdot\vec B) - \vec r\,(\vec j\cdot\vec B)\right]$$

$$\vec{N} = \frac{1}{2c}\int d^3r \left[\vec{j}\,(\vec{r}\cdot\vec{B}) - \vec{r}\,(\vec{j}\cdot\vec{B})\right]$$

$$\cancel{= \vec{m}\times\vec{B}}$$

$$= \frac{1}{2c}\int d^3r\,(\vec{r}\times\vec{j})\times\vec{B}$$

$$\boxed{\vec{N} = \vec{m}\times\vec{B}}$$

Electrostatic energy of interaction

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r \; E^2$$

Suppose the charge density ρ that produces $\vec{E}$ can be broken into two pieces, $\rho = \rho_1 + \rho_2$ with $\vec{E} = \vec{E}_1 + \vec{E}_2$ where $\vec{\nabla}\cdot\vec{E}_1 = 4\pi\rho_1$ and $\vec{\nabla}\cdot\vec{E}_2 = 4\pi\rho_2$

Then

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r \left[\underbrace{E_1^2}_{\text{"self-energy" of } \rho_1} + \underbrace{E_2^2}_{\text{"self energy" of } \rho_2} + \underbrace{2\vec{E}_1\cdot\vec{E}_2}_{\text{"interaction" energy of } \rho_1 \text{ with } \rho_2} \right]$$

$$\mathcal{E}_{int} = \frac{1}{4\pi}\int d^3r \; \vec{E}_1\cdot\vec{E}_2$$

$$= \int d^3r \; \rho_1 \Phi_2 = \int d^3r \; \rho_2 \Phi_1$$

where $\vec{E}_1 = -\vec{\nabla}\Phi_1$, $\vec{E}_2 = -\vec{\nabla}\Phi_2$, by similar manipulations as earlier

integrals are over <u>all</u> space

Apply to the interaction energy of a dipole in an external $\vec{E}$ field

$$\mathcal{E}_{int} = \int d^3r \; \underbrace{\rho_1}_{\text{charge distribution of dipole}} \; \underbrace{\Phi_2}_{\text{potential of external } \vec{E} \text{ field}}$$

Assuming ϕ_2 varys slowly on length scale of ρ_1 then we can expand $\phi_2(\vec{r}) = \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0)\cdot\vec{\nabla}\phi_2(\vec{r}_0)$ where $\vec{r}_0$ is the center of mass or any other convenient reference position within ρ_1

$$\mathcal{E}_{int} = \int d^3r\, \rho_1(\vec{r})\left[\phi_2(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla}\phi_2(\vec{r}_0)\right]$$

$$= q\,\phi_2(\vec{r}_0) + \left[\int d^3r\, \rho_1(\vec{r})(\vec{r}-\vec{r}_0)\right]\cdot\vec{\nabla}\phi_2(\vec{r}_0)$$

$$= q\,\phi_2(\vec{r}_0) - \vec{p}\cdot\vec{E}$$

where q is total charge in ρ_1 and $\vec{p}$ is dipole moment with respect to $\vec{r}_0$ $\vec{E} = -\vec{\nabla}\phi_2$ is external $\vec{E}$-field

For a neutral charge distribution $q = 0$, and $\vec{p}$ is independent of the origin about which it is computed, so

$$\boxed{\mathcal{E}_{int} = -\vec{p}\cdot\vec{E}}$$

← does not include the energy needed to make the dipole or to make $\vec{E}$.

$\mathcal{E}_{int}$ is lowest when $\vec{p} \parallel \vec{E}$

⇒ in thermal ensemble, dipoles tend to allign parallel to an applied $\vec{E}$.

Energy of magnetic dipole in external field

We had that the force on the dipole was

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

if we regard this force as coming from the gradient of a potential energy U then $\vec{F} = -\vec{\nabla}U \Rightarrow$

$$\boxed{U = -\vec{m}\cdot\vec{B}}$$

or equivalently, energy = work done to move dipole into position from ∞

$$W = -\int_{\infty}^{\vec{r}} \vec{F}\cdot d\vec{l} = -\int_{\infty}^{\vec{r}_0} \vec{\nabla}(m\cdot\vec{B})\cdot d\vec{l} = -\vec{m}\cdot\vec{B}(\vec{r})$$

This is the correct energy to use in cases where $\vec{m}$ is due to intrinsic magnetic moments of atom or molecule - say from electron or nuclear spin. For a thermal ensemble magnetic moments tend to allign $\parallel$ to $\vec{B}$.

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we did it the the energy of an electric dipole in an electric field...

Magnetostatic energy of interaction

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\; B^2$$

Suppose current $\vec{J}$ that produces $\vec{B}$ can be divided $\vec{J} = \vec{J}_1 + \vec{J}_2$ with $\vec{B} = \vec{B}_1 + \vec{B}_2$ where $\vec{\nabla}\times\vec{B}_1 = \frac{4\pi}{c}\vec{J}_1$ and $\vec{\nabla}\times\vec{B}_2 = \frac{4\pi}{c}\vec{J}_2$. then

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\left[\underbrace{B_1^2}_{\text{self energy of }\vec{J}_1} + \underbrace{B_2^2}_{\text{self energy of }\vec{J}_2} + \underbrace{2\vec{B}_1\cdot\vec{B}_2}_{\text{interaction energy of }\vec{J}_1\text{ with }\vec{J}_2}\right]$$

$$\mathcal{E}_{int} = \frac{1}{4\pi}\int d^3r\;\vec{B}_1\cdot\vec{B}_2$$

$$= \frac{1}{c}\int d^3r\;\vec{J}_1\cdot\vec{A}_2 = \frac{1}{c}\int d^3r\;\vec{J}_2\cdot\vec{A}_1$$

where $\vec{B}_1 = \vec{\nabla}\times\vec{A}_1$, $\vec{B}_2 = \vec{\nabla}\times\vec{A}_2$, by similar manipulations as earlier

integrals are over all space

Apply to the interaction energy of a magnetic dipole in an external $\vec{B}$ field.

$$\mathcal{E}_{int} = \frac{1}{c}\int d^3r\;\underbrace{\vec{J}_1}_{\text{current distribution of dipole}}\cdot\underbrace{\vec{A}_2}_{\text{vector potential of external }\vec{B}\text{ field}}$$

Assuming $\vec{A}$ varies slowly on length scale of $\vec{j}$, then expand $A_i(\vec{r}) = A_i(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla}A_i(\vec{r}_0)$

$$\mathcal{E}_{int} = \frac{1}{c}\int d^3r\, \vec{j}_1 \cdot \vec{A}(\vec{r}_0)$$

$$+ \frac{1}{c}\int d^3r \sum_{i,j} j_{1i}\,(r-r_0)_j\, \partial_j A_i(\vec{r}_0)$$

~~shift origin so origin at $\vec{r}_0$ ~ $\vec{r}$ now measures distance~~

From ~~magnetostatic~~ computation of magnetic dipole moment we had $\int d^3r\, \vec{j} = 0$ for magnetostatics

$\Rightarrow$ 1st term above vanishes. So does the piece of 2nd term $\left(\int d^3r\, j_{1i}\right) r_{0j}\, \partial_j A_i(\vec{r}_0)$

We are left with

$$\mathcal{E}_{int} = \left[\frac{1}{c}\int d^3r\; j_{1i}\, r_j\right]\partial_j A_i(\vec{r}_0)$$

summation over repeated indices is implied

From computation of magnetic dipole approx we had

$$\int d^3r\, j_{1i}\, r_j = -\int d^3r\, j_{1j}\, r_i$$

$$= \frac{1}{2}\int d^3r\left[j_{1i}r_j - j_{1j}r_i\right]$$

$$= \frac{1}{2}\,\varepsilon_{kij}\int d^3r\,(\vec{j}\times\vec{r})_k$$

Recall: $\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r}\times\vec{j}$

$$\Rightarrow \frac{1}{c}\int d^3r\; j_{1j}\, r_i = -\,\varepsilon_{kij}\, m_k \leftarrow \text{mag dipole moment}$$

$$\mathcal{E}_{int} = -m_k \varepsilon_{kij} \partial_j A_i = m_k \varepsilon_{kji} \partial_j A_i$$

$$= \vec{m} \cdot (\vec{\nabla} \times \vec{A}) = \vec{m} \cdot \vec{B} = \mathcal{E}_{int}$$

This is opposite in sign to what we found earlier!

Why the difference?

① When we integrate the work done against the magnetostatic force to move $\vec{m}$ into position from infinity we found the energy

$$U = -\vec{m} \cdot \vec{B}$$

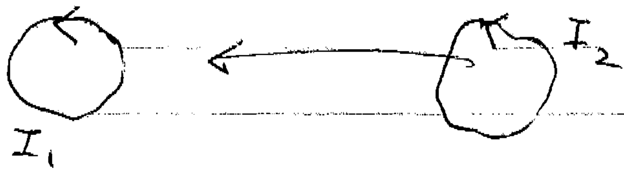
② When we compute the interaction energy from

$$\mathcal{E}_{int} = \frac{1}{c} \int d^3r\, \vec{j}_1 \cdot \vec{A}_2 = \frac{1}{c^2} \int d^3r \int d^3r' \frac{\vec{j}_1(\vec{r}) \cdot \vec{j}_2(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

we found the energy $\mathcal{E}_{int} = +\vec{m} \cdot \vec{B}$

To see which is correct, let us consider computing the interaction energy ② directly via method ①.

Consider two loops with currents I_1 and I_2
What is the work done to move loop 2 in from infinity to its final position with respect to loop 1?



Magnetostatic force on loop 2 due to loop 1 is

$$\vec{F} = \frac{I_2}{c} \oint_2 d\vec{l}_2 \times \vec{B}_1$$ Lorentz force

$\vec{B}_1$ is magnetic field from loop 1

$$\vec{B}_1(\vec{r}) = \frac{I_1}{c} \oint_1 d\vec{l}_1 \times \frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$ Biot-Savart law

$$F = \frac{I_1 I_2}{c^2} \oint_2 \oint_1 d\vec{l}_2 \times \frac{(d\vec{l}_1 \times (\vec{r}_2-\vec{r}_1))}{|\vec{r}_2-\vec{r}_1|^3}$$

use triple product rule

$$d\vec{l}_2 \times \left[d\vec{l}_1 \times (\vec{r}_2-\vec{r}_1)\right] = d\vec{l}_1 \left[d\vec{l}_2 \cdot (\vec{r}_2-\vec{r}_1)\right] - (\vec{r}_2-\vec{r}_1)(d\vec{l}_1 \cdot d\vec{l}_2)$$

from the 1st term

$$\oint_2 d\vec{l}_2 \cdot \frac{(\vec{r}_2-\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|^3} = -\oint_2 d\vec{l}_2 \cdot \vec{\nabla}_2 \left(\frac{1}{|\vec{r}_2-\vec{r}_1|}\right) = 0$$

as integral of gradient around closed loop always vanishes!

So

$$\vec{F} = -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{\ell}_1 \cdot d\vec{\ell}_2 \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3}$$

write $\vec{r}_2 = \vec{R} + \delta\vec{r}_2$ where $\vec{R}$ is center of loop 2

use $\dfrac{\vec{R} + \delta\vec{r}_2 - \vec{r}_1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|^3} = -\vec{\nabla}_R \left(\dfrac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$

$$\vec{F} = \frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{\ell}_1 \cdot d\vec{\ell}_2 \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

to move loop 2 we need to apply a mechanical force equal and opposite to the above magnetostatic force. Therefore the work we do in moving loop 2 from infinity to its final position at $\vec{R}_0$ is

$$W_{mech} = -\int_\infty^{\vec{R}_0} \vec{F} \cdot d\vec{R} = -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 d\vec{\ell}_1 \cdot d\vec{\ell}_2 \int_\infty^{\vec{R}_0} d\vec{R} \cdot \vec{\nabla}_R \left(\frac{1}{|\vec{R} + \delta\vec{r}_2 - \vec{r}_1|} \right)$$

$$= -\frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|\vec{r}_2 - \vec{r}_1|} \qquad \text{where } \vec{r}_2 = \vec{R}_0 + \delta\vec{r}_2$$

$$= -\frac{1}{c^2} \int d^3r_1 \int d^3r_2 \frac{\vec{j}_1(\vec{r}_1) \cdot \vec{j}_2(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|}$$

Note the minus sign!

this is just the negative of the interaction energy!!

$$= -M_{12} I_1 I_2$$

mutual inductance

Why the minus sign!

The minus sign we have here is the same minus sign we got when we found $U = -\vec{m}\cdot\vec{B}$ by integrating the force on the magnetic dipole.

Why dont we get $+\frac{1}{c^2}\int d^3r_1\, d^3r_2\, \frac{\vec{j}_1(r_1)\cdot\vec{j}_2(r_2)}{|\vec{r_2}-\vec{r_1}|}$

with the <u>plus</u> sign we expect from $\mathcal{E} = \frac{1}{8\pi}\int d^3r\, B^2$?

<u>Answer:</u> we have left something out!

<u>Faraday's law</u> - when we move loop 2, the magnetic flux through loop 2 changes. This $\frac{d\Phi}{dt}$ creates an emf $= \oint d\vec{l}\cdot\vec{E}$ around the loop that would tend to change the current in the loop. If we are to keep the current fixed at constant I_2 then there must be a battery in the loop that does work to counter this induced emf ("electromotive force"). Similarly, the flux through loop 1 is changing and a battery does work to keep I_1 constant. We need to add this work done by the batteries to the mechanical work computed above.

emf induced in loop 1 $\mathcal{E}_1 = \oint_1 d\vec{l}_1\cdot\vec{E}_2$
emf induced in loop 2 $\mathcal{E}_2 = \oint_2 d\vec{l}_1\cdot\vec{E}_1$
} integrations in direction of current

Faraday $\mathcal{E}_1 = \frac{-d\Phi_1}{c\,dt}$ $\quad\Phi_1$ = flux through loop 1

$\mathcal{E}_2 = \frac{-d\Phi_2}{c\,dt}$ $\quad\Phi_2$ = flux through loop 2

To keep the current constant, the batteries need to provide an emf that counters these Faraday induced emf's. The work done by the batteries per unit time is therefore

$$\frac{dW_{battery}}{dt} = -\mathcal{E}_1 I_1 - \mathcal{E}_2 I_2$$

(check units: $\mathcal{E} I$ is $[\text{length}]\cdot[E]\cdot[q/s]$
$= [\text{length}]\cdot[\text{force}/s]$
$= \text{energy}/s$)

$$\frac{dW_{battery}}{dt} = \frac{d\Phi_1}{c\,dt} I_1 + \frac{d\Phi_2}{c\,dt} I_2$$

$$W_{battery} = \int_0^T dt \left(\frac{d\Phi_1}{c\,dt} I_1 + \frac{d\Phi_2}{c\,dt} I_2 \right)$$

where $t=0$ loop 2 is at infinity
$t=T$ loop 2 is at final position
I_1, I_2 kept constant as loop moves

$$W_{battery} = \frac{1}{c}\Phi_1 I_1 + \frac{1}{c}\Phi_2 I_2$$

where Φ_1 and Φ_2 are fluxes in final position, and we assumed that fluxes $=0$ at infinity

$$\Phi_1 = cM_{12} I_2$$

$$\Phi_2 = cM_{21} I = cM_{12} I_1 \quad \text{as } M_{12} = M_{21}$$

$$\Rightarrow W_{battery} = 2 M_{12} I_1 I_2$$

add this to the mechanical work

$$W_{total} = W_{mech} + W_{battery} = -M_{12} I_1 I_2 + 2 M_{12} I_1 I_2$$

$$= M_{12} I_1 I_2 = +\frac{1}{c^2} \int d^3r_1 \, d^3r_2 \, \frac{\vec{J}_1(r_1) \cdot \vec{J}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$

we get back the correct interaction energy!

<u>Conclusion</u> : The magnetostatic interaction energy $\frac{1}{c^2} \int d^3r_1 \, d^3r_2 \, \frac{\vec{J}_1(\vec{r}_1) \cdot \vec{J}_2(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$

includes the work done to maintain the currents stationary as the current distributions move.

When we computed the interaction energy of a <u>current loop</u> dipole $\vec{m}$ and find

$$\mathcal{E}_{int} = +\vec{m} \cdot \vec{B}$$

this includes the energy needed to maintain the constant current producing the constant $\vec{m}$

When we integrated the force on the dipole to find the potential energy

$$U = -\vec{m} \cdot \vec{B}$$

this <u>did not</u> include the energy needed to maintain ~~the~~ constant current that creates $\vec{m}$. This is the correct energy expression to use when $\vec{m}$ comes from <u>intrinsic</u> <u>magnetic</u> <u>moments</u> due to particles intrinsic spin, which cannot be viewed as arising from a current loop!

Electromagnetic Waves in a Vacuum

No sources $\vec{J}=0$, $\rho=0$

1) $\vec{\nabla}\cdot\vec{E}=0$ 3) $\vec{\nabla}\cdot\vec{B}=0$

2) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{c\,\partial t}$ 4) $\vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

$$\nabla\times(2)\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{0 \text{ by } (1)}-\nabla^2\vec{E}=-\frac{\partial}{c\,\partial t}(\vec{\nabla}\times\vec{B})$$

$$-\nabla^2E=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})=-\frac{\partial}{c\,\partial t}\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)$$

$$\nabla^2\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0$$

Similarly

$$\nabla^2\vec{B}-\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}=0$$

} wave equation
wave speed is c.

Note: in MKS units, above wave equation looks like

$$\nabla^2\vec{E}-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}=0$$

It was noticed that the speed of electromagnetic wave,

$\frac{1}{\sqrt{\varepsilon_0\mu_0}}=3\times10^8\,\text{m/s}$ was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic wave

Harmonic

Plane waves

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$\vec{B}(\vec{r},t) = \mathrm{Re}\left[\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

complex exponential form

$\vec{k}$ is wave vector

ω is angular frequency

$\nu = \omega/2\pi$ is frequency

$T = 1/\nu$ is period

$\lambda = \dfrac{2\pi}{|k|}$ is wavelength

$|\vec{E}_k|$, $|\vec{B}_k|$ is amplitude

$\vec{E}(\vec{r}+\lambda\hat{k},t) = \vec{E}(\vec{r},t)$ periodic in space with period λ

$\vec{E}(\vec{r},t+T) = \vec{E}(r,t)$ periodic in time with period T

"plane wave" $\Rightarrow$ $\vec{E}(\vec{r},t)$ is constant in space on planes with normal $\hat{n} \parallel \vec{k}$.

properties of EM plane waves

$$\vec{\nabla}\cdot\vec{E}=0 \implies \mathrm{Re}\left[\vec{E}_k\cdot\vec{\nabla}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$
$$= \mathrm{Re}\left[i\vec{E}_k\cdot\vec{k}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$
$$\Rightarrow \vec{E}_k\cdot\vec{k}=0$$

amplitude is orthogonal to $\vec{k}$

$$\vec{\nabla}\cdot\vec{B}=0 \rightarrow \vec{B}_k\cdot\vec{k}=0$$

amplitude orthogonal to $\vec{k}$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \operatorname{Re}\left[\vec{\nabla}\times\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \operatorname{Re}\left[\frac{1}{c}\vec{E}_k\frac{\partial}{\partial t}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \operatorname{Re}\left[-\vec{B}_k\times\vec{\nabla}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \operatorname{Re}\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \operatorname{Re}\left[i\vec{k}\times\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \operatorname{Re}\left[-\frac{i\omega}{c}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$\Rightarrow \quad \vec{k}\times\vec{B}_k = -\frac{\omega}{c}\vec{E}_k$$

$$\vec{k}\times\vec{k}\times\vec{B}_k = -k^2\vec{B}_k = -\frac{\omega}{c}\vec{k}\times\vec{E}_k$$

$$\underline{\vec{B}_k = \frac{\omega}{ck^2}\vec{k}\times\vec{E}_k}$$

Finally

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\Rightarrow \operatorname{Re}\left[\vec{E}_k\nabla^2 e^{i(\vec{k}\cdot\vec{r}-\omega t)} - \frac{\vec{E}_k}{c^2}\frac{\partial^2}{\partial t^2}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \operatorname{Re}\left[\vec{E}_k(-k^2)e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \frac{\omega^2}{c^2}\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = 0$$

$$\Rightarrow \quad k^2 = \frac{\omega^2}{c^2}$$

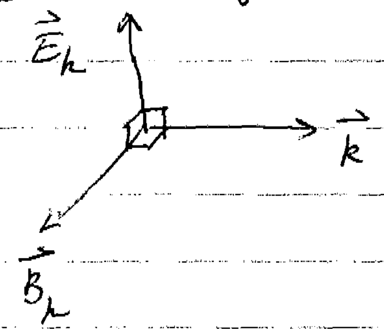
$$\boxed{\omega = \pm kc}$$ <u>dispersion relation</u>

insert in above

$$\vec{B}_k = \hat{k}\times\vec{E}_k \qquad \hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

$$\Rightarrow \quad |\vec{B}_k| = |\vec{E}_k|$$

Summary



$$\left.\begin{array}{l}\vec{E}_k \perp \vec{k} \\ \vec{B}_k \perp \vec{k}\end{array}\right\}\text{"transverse" polarization}$$

$$\vec{B}_k = \hat{k} \times \vec{E}_k$$

$$\omega^2 = c^2 k^2$$

$|\vec{B}_k| = |\vec{E}_k| \Rightarrow$ Lorentz force from plane EM wave on charge q is

$$q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$$

magnetic force is smaller factor $\left(\frac{v}{c}\right)$ as compared to electric force - can usually be ignored

Most general solution is a linear superposition of the above harmonic plane waves

$$\vec{E}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\, \vec{E}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \qquad \text{Fourier transform}$$

$$\vec{E}(\vec{r},t) \text{ is real} \Rightarrow \vec{E}_k^{\,*} = \vec{E}_{-k}$$

For dispersion relation $\omega^2 = c^2k^2$ we can write

$$\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot(\vec{r} - \vec{v}t)$$

where $\vec{v} = c\hat{k}$ is velocity of wave

If we ~~only~~ combine waves traveling in same direction $\hat{k}$, then

$$\vec{E}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\, \vec{E}_k\, e^{i\vec{k}\cdot(\vec{r}-\vec{v}t)} = \vec{E}(\vec{r}-\vec{v}t, 0)$$

The general plane wave solution of wave equation always has this property

$$\vec{E}(r,t) = \vec{E}(\vec{r}-\vec{v}t, 0)$$

If know $\vec{E}$ at $t=0$, then know $\vec{E}$ at all times t

Energy & momentum in EM wave

$$\left.\begin{aligned} \vec{E} &= Re\left[\vec{E}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t) \\ \vec{B} &= Re\left[\vec{B}_k e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = \hat{k}\times\vec{E}_k \cos(\vec{k}\cdot\vec{r}-\omega t) \end{aligned}\right\} \text{for real } \vec{E}_k$$

energy density $\quad u = \frac{1}{8\pi}\left(E^2 + B^2\right)$

$$= \frac{1}{8\pi}\left[E_k^2 + E_k^2\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{1}{4\pi} E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

Poynting vector

energy current $\quad \vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}$

$$= \frac{c}{4\pi}\left[\vec{E}_k \times (\hat{k}\times\vec{E}_k)\right]\cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$= \frac{c}{4\pi}\hat{k}\, E_k^2 \cos^2(\vec{k}\cdot\vec{r}-\omega t)$$

$$\vec{S} = c\, u\, \hat{k}$$

momentum density $\quad \vec{\Pi} = \frac{1}{c^2}\vec{S} = \frac{u}{c}\hat{k}$

$u = c\,|\vec{\Pi}|$ — energy momentum relation of photons!

For visible light $\lambda \sim 5\times 10^{-7}\,m \sim 5000\,Å$

$$T = \frac{\lambda}{c} = 1.6\times 10^{-15}\text{ sec}$$

most classical measurements on microscopic scales $\quad t \gg T, \quad \ell \gg \lambda$

measure average quantities

$$\langle u \rangle = \frac{1}{T}\int_0^T dt\ u = \frac{1}{8\pi} E_k^2 \qquad \text{as } \langle \cos^2\theta \rangle = 1/2$$

$$\langle \vec{S} \rangle = c \langle u \rangle \hat{k}$$

$$\langle \vec{\Pi} \rangle = \frac{1}{c} \langle u \rangle \hat{k}$$

intensity = average power per area transported by wave through surface with normal $\hat{n}$

$$I = \langle \vec{S} \rangle \cdot \hat{n}$$

Electromagnetic waves in matter

Macroscopic Maxwell equations with no sources
("free" charge and current vanishes)

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

linear materials

$$\vec{B} = \mu\vec{H}$$
$$\vec{D} = \epsilon\vec{E}$$

if μ and ϵ were simply constants then the above would become

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

Then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2 E = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$$
$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\mu\epsilon}{c}\frac{\partial E}{\partial t}\right)$$
$$= -\frac{\mu\epsilon}{c^2}\frac{\partial^2 E}{\partial t^2}$$

wave equation with wave speed $\frac{c}{\sqrt{\mu\epsilon}} < c$

This would be very much as for waves in a vacuum, except for the following minor changes:

$$\omega^2 = \frac{c^2 k^2}{\mu \epsilon}$$ dispersion relation changed by constant factor

$$\vec{E}_k \perp \vec{k}$$
$$\vec{B}_k \perp \vec{k}$$

$$i \vec{k} \times \vec{E}_k = \frac{i\omega}{c} \vec{B}_k$$

$$\frac{c|\vec{k}|}{\omega} \hat{k} \times \vec{E}_k = \vec{B}_k$$

$$\Rightarrow \sqrt{\mu\epsilon}\, \hat{k} \times \vec{E}_k = \vec{B}_k \qquad |\vec{B}_k| > |\vec{E}_k|$$

wave speed $v = \frac{c}{\sqrt{\mu \epsilon}} < c$

In general however things are much more complicated because ϵ cannot be viewed as a constant when considering time varying behavior!