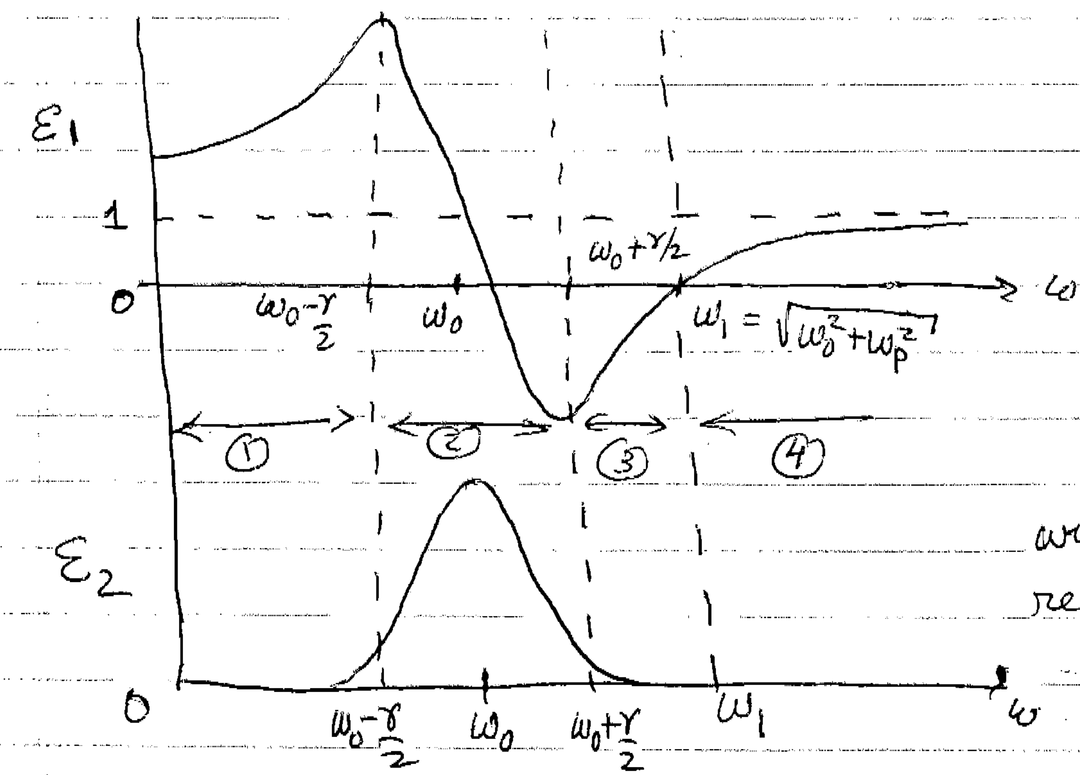
For our simple model: $\varepsilon = 1 + 4\pi\chi \simeq 1 + 4\pi M\alpha$

$$\varepsilon(\omega) = 1 + \frac{4\pi M e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$\varepsilon_1 = 1 + \frac{4\pi M e^2}{m} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\varepsilon_2 = \frac{4\pi M e^2}{m} \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

Define $\omega_p = \sqrt{\frac{4\pi M e^2}{m}}$ the "plasma frequency"




width $\sim \gamma$
resonance at $\bar{\omega} \approx \omega_0$

$$k = k_1 + ik_2 = \pm \frac{\omega}{c} \sqrt{\mu} \sqrt{\varepsilon_1 + i\varepsilon_2}$$

$$k^2 = k_1^2 - k_2^2 + 2ik_1k_2 = \frac{\omega^2}{c^2} \mu (\varepsilon_1 + i\varepsilon_2)$$

equate real and imaginary pieces and solve for $k_1$ and $k_2$

$$k_1 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_2 = \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\sqrt{\varepsilon_1^2+\varepsilon_2^2} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

<u>Regions of different behavior</u>

<u>Regions ① and ④</u> — transparent propagation

$\varepsilon_1 > 0$ , $\varepsilon_1 \gg \varepsilon_2$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\varepsilon_1 + \frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu\varepsilon_1} \quad + \text{ small correction}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_1\left(1+\frac{1}{2}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right) - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{4}\frac{\varepsilon_2^2}{\varepsilon_1}\right]^{1/2} = k_1\left(\frac{\varepsilon_2}{2\varepsilon_1}\right) \ll k_1$$

So $k_2 \ll k_1$ small attenuation

$\Rightarrow$ medium is transparent

<u>Note</u>: $v_p = \frac{\omega}{k_1} = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon_1\mu}}$

in region ①, $\varepsilon_1 > 1 \Rightarrow v_p < c$

in region ④, $\varepsilon_1 < 1 \Rightarrow v_p > c$ !

but $v_g < c$ always!

Region (2) $\omega \simeq \omega_0$ resonant absorption

$$\varepsilon_2 \simeq \frac{\omega_p^2}{\omega_0 \gamma} = \left(\frac{\omega_p}{\omega_0}\right)^2 \left(\frac{\omega_0}{\gamma}\right) \gg 1$$ for a sharp resonance with $\gamma \ll \omega_0$

$$\varepsilon_1 \simeq 1$$

So $\varepsilon_2 \gg \varepsilon_1$

$$k_1 \simeq \pm \frac{\omega\sqrt{\mu}}{c}\left[\frac{1}{2}\varepsilon_2\left(1+\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right)+\frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2+\frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2}+\frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\tfrac{1}{2}\varepsilon_2}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2\left(1+\frac{1}{2}\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2\right)-\frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}\varepsilon_2+\frac{1}{4}\frac{\varepsilon_1^2}{\varepsilon_2}-\frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu}\sqrt{\tfrac{1}{2}\varepsilon_2}$$

$k_1 \approx k_2$ strong attenuation

wave excites atoms at resonance $\Rightarrow$ large atomic displacements $\Rightarrow$ media absorbs most energy from the wave $\Rightarrow$ wave decays rapidly, decreases factor $\frac{1}{e^{2\pi}}$ within one wavelength of propagation.

Region ③   $\varepsilon_1 < 0, \quad |\varepsilon_1| \gg \varepsilon_2$

total reflection

width of region ③ is

$$\omega_1 - \omega_0 = \sqrt{\omega_0^2 + \omega_p^2} - \omega_0 \sim \omega_p \sim \sqrt{N}$$

increases with atomic density   as $\omega_p \gg \omega_0$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\varepsilon_1| + \frac{1}{4}\frac{\varepsilon_2^2}{|\varepsilon_1|} + \frac{1}{2}\varepsilon_1\right]^{1/2}$$

cancel as $|\varepsilon_1| = -\varepsilon_1$

$$k_1 \simeq \pm \frac{\omega}{c}\sqrt{\mu|\varepsilon_1|}\,\frac{\varepsilon_2}{2|\varepsilon_1|}$$

$$k_2 \simeq \pm \frac{\omega}{c}\sqrt{\mu}\left[\frac{1}{2}|\varepsilon_1| + \frac{1}{4}\frac{\varepsilon_2^2}{|\varepsilon_1|} - \frac{1}{2}\varepsilon_1\right]^{1/2}$$

$$\simeq \pm \frac{\omega}{c}\sqrt{\mu|\varepsilon_1|}$$

$$\frac{k_2}{k_1} = \frac{2|\varepsilon_1|}{\varepsilon_2} \gg 1$$

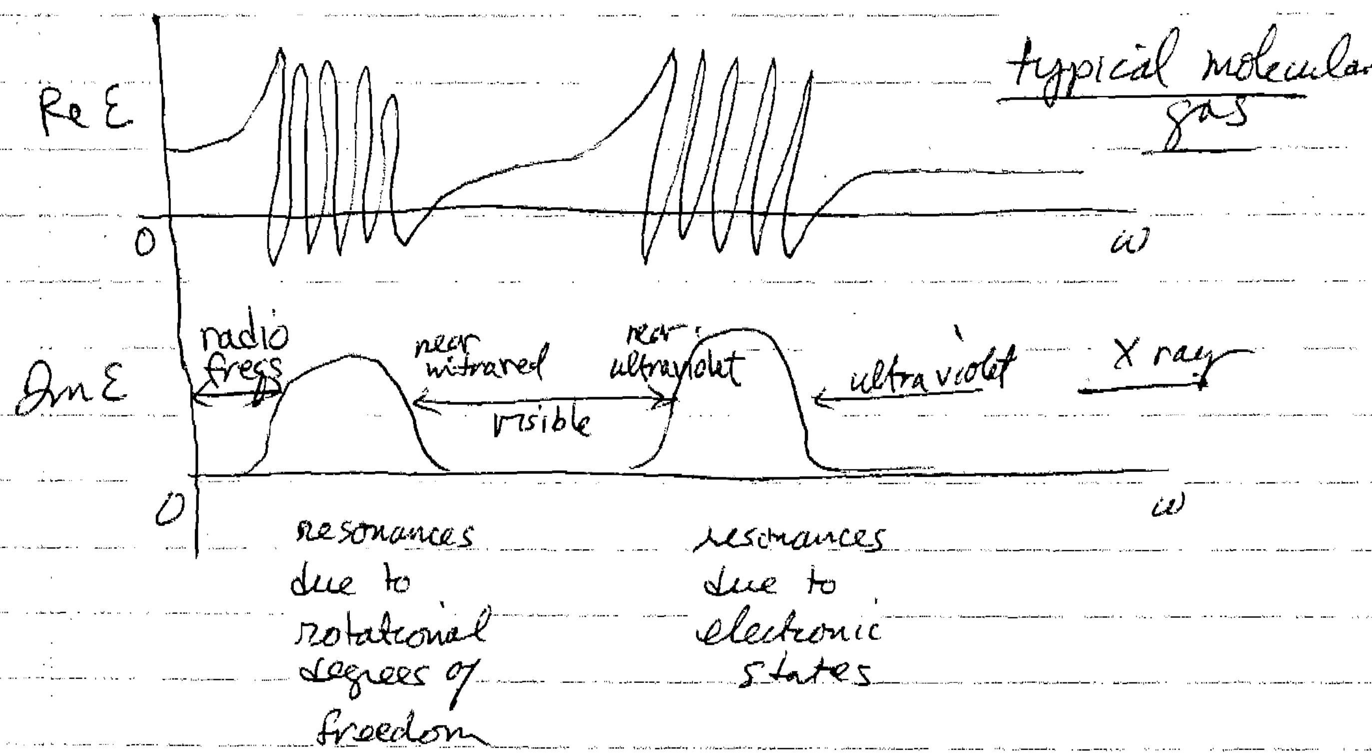
wave vector is almost pure imaginary
wave decays exponentially to zero in much less than one wavelength.

we will see this corresponds to total reflection
Since $\omega \gg \omega_0$, we are not at resonance so material is not absorbing much energy from wave. The strong attenuation is due to the destructive interference between the wave and the induced fields of the polarized atoms

Our single model had a single resonance at $\omega_0$. A more realistic model for molecules has many bands of resonances due to rotational, vibrational, and electronic modes of excitation

$$\varepsilon(\omega) = 1 + \omega_p^2 \sum_i \frac{f_i}{\omega_i^2 - \omega^2 - i\omega\gamma_i}$$

where $\hbar\omega_i$ are spacings between energy levels with allowed electric dipole transitions



$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

$$= 4.4\times10^{16}\sqrt{\frac{n}{n_A}}\ \sec^{-1}, \quad n_A = 6\times10^{23}/cm^3$$

$$\Rightarrow \hbar\omega_p = 185\sqrt{\frac{n}{n_A}}\ eV$$

For $H_2O$ $\quad \frac{n}{n_A} \sim 0.05$

$$\hbar\omega_p \sim 40\ eV$$

For typical metal $\frac{n}{n_A} \sim 0.1$

$$\hbar\omega_p \sim 58\ eV$$

compared to $\hbar\omega_0 \sim eV$

# EM waves in Conductors

Conduction electrons are mobil, not bound
$\Rightarrow$ we have to include the $\vec{J}_f$ and $\rho_f$ from them.

Simple classical model for electron motion – "Drude" Model

$$m\ddot{\vec{r}} = -e\vec{E}(t) - \frac{m}{\tau}\dot{\vec{r}}$$

external E field

damping force due to collisions
$\tau$ is "relaxation time"

$\vec{E} = \vec{E}_\omega e^{-i\omega t} \Rightarrow \vec{r} = \vec{r}_\omega e^{-i\omega t}$ solution

plug in to get

$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\vec{r}_\omega = -\frac{e}{m}\vec{E}_\omega \Rightarrow \vec{r}_\omega = \frac{e}{m}\frac{1}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega$$

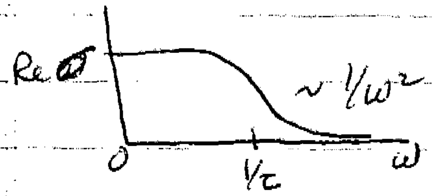
current is $\vec{J}_f = -en\dot{\vec{r}}_\omega = -en(-i\omega)\vec{r}_\omega$

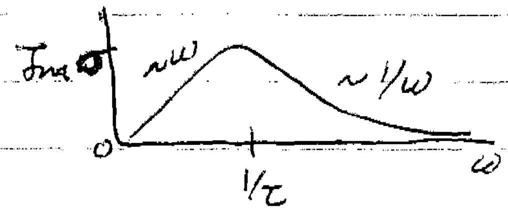
$n$ density of electrons

$$\vec{J}_f = \frac{ne^2}{m}\frac{i\omega}{\omega^2 + \frac{i\omega}{\tau}}\vec{E}_\omega = \frac{ne^2\tau}{m}\frac{1}{1 - i\omega\tau}\vec{E}_\omega$$

$$\vec{J}_f = \sigma(\omega)\vec{E}_\omega$$

conductivity $\boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\frac{1}{1 - i\omega\tau}}$




$$\text{Re}\,\sigma = \frac{\sigma_0}{1+\omega^2\tau^2} \qquad \text{Im}\,\sigma = \sigma_0\frac{\omega\tau}{1+\omega^2\tau^2}$$

$$\sigma_0 = \sigma(0) = \frac{ne^2\tau}{m}$$ dc conductivity

Charge density $\rho_f$ given by charge conservation law.
For plane waves

$$\rho_f = \rho_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \vec{j}_f = \vec{j}_\omega e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\frac{\partial \rho_f}{\partial t} = -\vec{\nabla}\cdot\vec{j}_f \Rightarrow -i\omega\rho_\omega = -i\vec{k}\cdot\vec{j}_\omega$$

$$\rho_\omega = \frac{\vec{k}\cdot\vec{j}_\omega}{\omega} = \frac{\sigma(\omega)\,\vec{k}\cdot\vec{E}_\omega}{\omega}$$

<u>Maxwell Equations</u>

1) $\vec{\nabla}\cdot\vec{D} = 4\pi\rho_f$ 2) $\vec{\nabla}\cdot\vec{B} = 0$

3) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ 4) $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j}_f + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$

Assume $\vec{H} = \vec{B}/\mu$, $\mu$ constant

$\vec{D}_\omega = \varepsilon_b(\omega)\vec{E}_\omega$ $\varepsilon_b(\omega)$ is dielectric function from the bound charges

$\vec{j}_\omega = \sigma(\omega)\vec{E}_\omega$ $\sigma(\omega)$ is conductivity from free charges

$\rho_\omega = \frac{\sigma}{\omega}\vec{k}\cdot\vec{E}_\omega$