We will see that this situation in general corresponds to an <u>elliptically polarized</u> wave!

<u>General case</u> E_1 and E_2 are complex constants

write $E_1\hat{e}_1 + E_2\hat{e}_2 \equiv \vec{u}\, e^{i\psi}$

where ψ is chosen so that $\vec{u}\cdot\vec{u}$ is <u>real</u>

– one can always do this since $\vec{u}\cdot\vec{u} = (E_1^2+E_2^2)\,e^{-2i\psi}$

so 2ψ is just the phase of the complex $E_1^2+E_2^2$

$\vec{u}$ is a complex vector $\Rightarrow$ $\vec{u} = \vec{u}_a + i\vec{u}_b$

with $\vec{u}_a$ and $\vec{u}_b$ real vectors

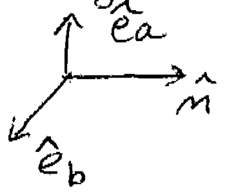
Since $\vec{u}\cdot\vec{u}$ is real $\Rightarrow$ $\vec{u}_a\cdot\vec{u}_b = 0$

so $\vec{u}_a \perp \vec{u}_b$ orthogonal

Let $\hat{e}_a$ be the unit vector in direction of $\vec{u}_a$

so $\vec{u}_a = u_a\,\hat{e}_a$ with $u_a = |\vec{u}_a|$

Let $\hat{e}_b = \hat{n}\times\hat{e}_a$ so that $\{\hat{n}, \hat{e}_a, \hat{e}_b\}$ are a right handed coordinate system



Then $\vec{u}_b = \pm\, u_b\,\hat{e}_b$ where $u_b = |\vec{u}_b|$

since $\vec{u}_b \perp \vec{u}_a$ and both are $\perp$ to $\hat{n}$.

It is (+) if $\vec{u}_b$ is parallel to $\hat{e}_b$ and it is (−) if $\vec{u}_b$ is antiparallel to $\hat{e}_b$.

In this representation we have

$$\vec{E}(\vec{r},t) = Re\left\{ \vec{u}\, e^{i\psi} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \right\}$$

$$= e^{-k_2 \hat{n}\cdot\vec{r}}\, Re\left\{ u_a \hat{e}_a e^{i(k_1 \hat{n}\cdot\vec{r} - \omega t + \psi)} \pm u_b \hat{e}_b (+i) e^{i(k_1 \hat{n}\cdot\vec{r} - \omega t + \psi)} \right\}$$

$$= e^{-k_2 \hat{n}\cdot\vec{r}} \left\{ u_a \hat{e}_a \cos(\Phi+\psi) \mp u_b \hat{e}_b \sin(\Phi+\psi) \right\}$$

where we write $\Phi \equiv k_1 \hat{n}\cdot\vec{r} - \omega t$

Let's define
$$e^{-k_2 \hat{n}\cdot\vec{r}}\, u_a \rightarrow u_a$$
$$e^{-k_2 \hat{n}\cdot\vec{r}}\, u_b \rightarrow u_b$$

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of $\vec{E}$.

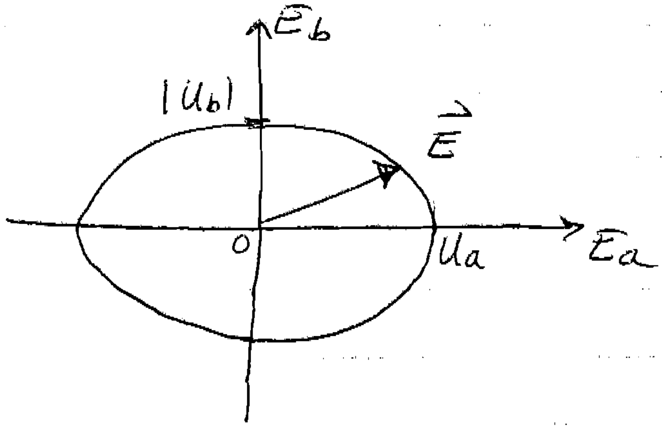
Then define E_a and E_b as the components of $\vec{E}$ in the directions $\hat{e}_a$ and $\hat{e}_b$ respectively.

$$E_a = u_a \cos(\Phi+\psi)$$
$$E_b = \mp u_b \sin(\Phi+\psi)$$

This then gives

$$\left(\frac{E_a}{u_a}\right)^2 + \left(\frac{E_b}{u_b}\right)^2 = \cos^2(\Phi+\psi) + \sin^2(\Phi+\psi) = 1$$

This is just the equation for an ellipse with semiaxes of lengths U_a and U_b, oriented in the directions of $\hat{e}_a$ and $\hat{e}_b$.



$\Rightarrow$ At a fixed position $\vec{r}$, the tip of the vector $\vec{E}$ will trace out the above ellipse as the time increases by one period of oscillation $2\pi/\omega$.

For (+), ie $\vec{U}_b = U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>counterclockwise</u> as t increases

For (−), ie $\vec{U}_b = -U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse <u>clockwise</u> as t increases

Such a wave is said to be <u>elliptically polarized</u>

<u>Special cases</u>

① $U_a = 0$ or $U_b = 0$
the wave is <u>linearly polarized</u>

(2) $u_a = u_b$

The tip of $\vec{E}$ traces out a ~~circl~~ circle as t increases. The wave is <u>circularly polarized</u>.

The (+) case is said to have <u>right handed circular polarization</u>

The (−) case is said to have <u>left handed circular polarization</u>

One can define circular polarization basis vectors

$$\hat{e}_+ \equiv \frac{\hat{e}_a + i\hat{e}_b}{\sqrt{2}} \qquad \hat{e}_- \equiv \frac{\hat{e}_a - i\hat{e}_b}{\sqrt{2}}$$

with $\hat{e}_a$ and $\hat{e}_b$ orthogonal.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_+$ is right handed circularly polarized.

A wave with complex amplitude $\vec{E}_\omega = E\,\hat{e}_-$ is left handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, ie

$$\vec{E}_\omega = E_1\,\hat{e}_1 + E_2\,\hat{e}_2$$

One can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

$$\vec{U} = \vec{U}_a + i\vec{U}_b = U_a\hat{e}_a \pm iU_b\hat{e}_b$$

$$= \left(\frac{U_a + U_b}{\sqrt{2}}\right)\hat{e}_{\pm} + \left(\frac{U_a - U_b}{\sqrt{2}}\right)\hat{e}_{\mp}$$

(~~expand~~ substitute in for $\hat{e}_{\pm}$ and expand, to see that this is so)

$\Rightarrow$ An elliptically polarized wave can be written as a superposition of circularly polarized waves

As a special case of the above (if $U_a = 0$ or $U_b = 0$) a <u>linearly</u> polarized wave can always be written as a superposition of circularly polarized waves.

magnetic field

In the above general formulation we can write $\vec{H}$ as

$$\vec{H} = \frac{c}{\omega\mu} \mathrm{Re}\left\{ k\hat{m} \times \vec{U} e^{i\psi} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right\}$$

$$= \frac{c|k|}{\omega\mu} \mathrm{Re}\left\{ \hat{m} \times (U_a \hat{e}_a \pm i U_b \hat{e}_b) e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta + \psi)} \right\}$$

$$= \frac{c|k|}{\omega\mu} \mathrm{Re}\left\{ (U_a \hat{e}_b \mp i U_b \hat{e}_a) e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta + \psi)} \right\}$$

$$\vec{H} = \frac{c|k|}{\omega\mu} e^{-k_2 \hat{m}\cdot\vec{r}} \left[U_a \hat{e}_b \cos(\Phi + \psi + \delta) \pm U_b \hat{e}_a \sin(\Phi + \psi + \delta) \right]$$

we had for the electric field

$$\vec{E} = e^{-k_2 \hat{m}\cdot\vec{r}} \left[U_a \hat{e}_a \cos(\Phi + \psi) \mp U_b \hat{e}_b \sin(\Phi + \psi) \right]$$

Consider $\vec{E}\cdot\vec{H}$. From the above, with $\hat{e}_a \cdot \hat{e}_b = 0$, we get

$$\vec{E}\cdot\vec{H} = e^{-2k_2 \hat{m}\cdot\vec{r}} \frac{c|k|}{\omega\mu} U_a U_b (\pm 1) \left[\sin(\Phi + \psi + \delta)\cos(\Phi + \psi) - \cos(\Phi + \psi + \delta)\sin(\Phi + \psi) \right]$$

$$= e^{-2k_2 \hat{m}\cdot\vec{r}} \frac{c|k|}{\omega\mu} U_a U_b (\pm 1) \sin\delta$$

where in the last step we used $\sin A \cos B - \cos A \sin B = \sin(A - B)$

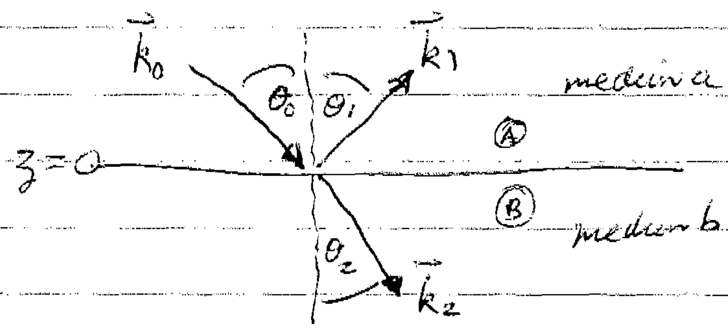
We see that $\vec{E}\cdot\vec{H} = 0$ only when

1) $\delta = 0$, ie the medium has no dissipation

or

2) $U_a = 0$ or $U_b = 0$, ie the wave is <u>linearly</u> polarized

Reflection + Transmission of waves at Interfaces



consider wave propagating from medium A into medium B.

for simplicity assume ϵ_a is real and positive, ϵ_b may be complex
μ_a and μ_b are real and constant

$\vec{k}_0$ is incident wave, θ_0 = angle of incidence
$\vec{k}_1$ is reflected wave, θ_1 = angle of reflection
$\vec{k}_2$ is the transmitted or "refracted" wave, θ_2 = angle of refraction

Let each wave be given by

$$\vec{F}_n(\vec{r},t) = \vec{F}_n\, e^{i(\vec{k}_n\cdot\vec{r} - \omega_n t)}$$

where $\vec{F}_n$ can be either $\vec{E}_n$ or $\vec{H}_n$ for the electric or magnetic component of the wave

boundary condition: tangential component $\vec{E}$ must be continuous at $z=0$. If $\hat{t}$ is a vector in xy plane, and we consider $\vec{r}=0$, then

$$\Rightarrow \hat{t}\cdot\vec{E}_0\, e^{-i\omega_0 t} + \hat{t}\cdot\vec{E}_1\, e^{-i\omega_1 t} = \hat{t}\cdot\vec{E}_2\, e^{-i\omega_2 t}$$

must be true for all time. Can only happen if

$\boxed{\omega_0 = \omega_1 = \omega_2 \equiv \omega}$ all frequencies are equal

Now consider the same boundary condition for $\vec{\rho}$ a position vector in the xy plane at $z=0$. Since ω's all equal we can cancel out the common $e^{-i\omega t}$ factors to get

$$\hat{t}\cdot\vec{E}_0\, e^{i\vec{k}_0\cdot\vec{\rho}} + \hat{t}\cdot E_1\, e^{i\vec{k}_1\cdot\vec{\rho}} = \hat{t}\cdot\vec{E}_2\, e^{i\vec{k}_2\cdot\vec{\rho}}$$

this must be true for all $\vec{\rho}$. Can only happen if the projections of the $\vec{k}_n$ in the xy plane are all equal

$$\boxed{\begin{aligned} k_{0x} &= k_{1x} = k_{2x} \\ k_{0y} &= k_{1y} = k_{2y} \end{aligned}}$$

only z components $\vec{k}$ vectors can be different

Choose coord system as in diagram so that all $\vec{k}$ vectors lie in the xz plane (y is out of page)

Since ϵ_a is real and positive, $\vec{k}_0$ and $\vec{k}_1$ are real vectors

$$k_{0x} = k_{1x} \Rightarrow |\vec{k}_0|\sin\theta_0 = |\vec{k}_1|\sin\theta_1$$

Since $k_0^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a$ and $k_1^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a$

then $|\vec{k}_0| = |\vec{k}_1|$ so $\sin\theta_0 = \sin\theta_1$

$$\boxed{\theta_0 = \theta_1}$$

angle of incidence = angle of reflection

If ϵ_b is also real and positive (B is transparent)
then $|\vec{k}_2|$ is real

$$k_{0x} = k_{2x} \Rightarrow |\vec{k}_0| \sin\theta_0 = |\vec{k}_2| \sin\theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_b$$

$$\Rightarrow \sqrt{\mu_a \epsilon_a} \sin\theta_0 = \sqrt{\mu_b \epsilon_b} \sin\theta_2$$

in terms of index of refraction $n = \frac{kc}{\omega} = \frac{\omega\sqrt{\mu\epsilon}}{c}\frac{c}{\omega}$

$$n = \sqrt{\mu\epsilon}$$

$$\Rightarrow n_a \sin\theta_0 = n_b \sin\theta_2$$

$$\boxed{\frac{\sin\theta_2}{\sin\theta_0} = \frac{n_a}{n_b}}$$

Snell's Law
true for all types of waves, not just EM waves

If $n_a > n_b$ then $\theta_2 > \theta_0$
In this case, when θ_0 is too large, we will have
$\frac{n_a}{n_b} \sin\theta_0 > 1$ and there will be no solution for θ_2
$\Rightarrow$ no transmitted wave

This is "total internal reflection" – wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

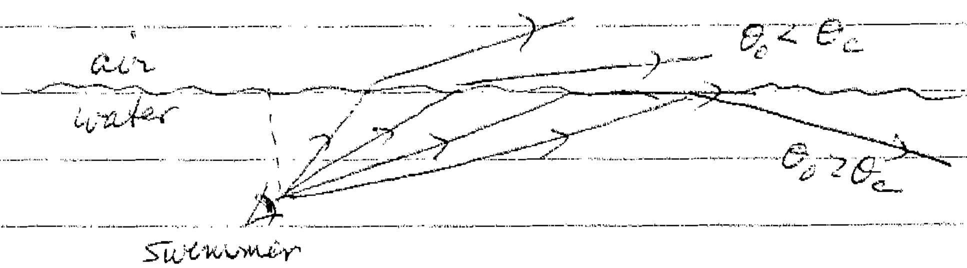
$$\frac{n_a}{n_b} \sin\theta_c = 1 \quad , \quad \boxed{\theta_c = \arcsin\left(\frac{n_b}{n_a}\right)}$$

$$\epsilon \sim 1 + 4\pi N \alpha \quad N \sim \text{density}$$

Since $n \approx \sqrt{\mu\epsilon}$ and ϵ grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

Examples: diamonds sparkle due to total internal reflection. Diamonds have large $n \Rightarrow$ small θ_c $\Rightarrow$ light bounces around inside many times before it can exit

Can also see total internal reflection when swimming under water



More general case: $\sqrt{\epsilon_b}$ is complex so $\vec{k}_2$ is complex

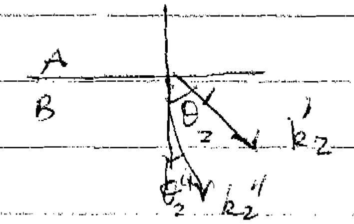
$$\vec{k}_2 = \vec{k}_2' + i\vec{k}_2'' \qquad k_2' = |\vec{k}_2'|, \quad k_2'' = |\vec{k}_2''|$$

($\vec{k}_2'$: real part; $\vec{k}_2''$: imaginary part)

Note $\vec{k}_2'$ and $\vec{k}_2''$ need not be in the same direction!

Condition $k_{0x} = k_{2x} \Rightarrow \begin{cases} k_{0x} = k_{2x}' \\ 0 = k_{2x}'' \end{cases}$ equate real and imaginary parts

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$
$$0 = k_2'' \sin\theta_2''$$



$$\Rightarrow \theta_2'' = 0 \qquad \vec{k}_2'' = k_2''\hat{z}$$

attenuation factor for the transmitted wave is $e^{-k_2'' z}$

$\Rightarrow$ planes of constant amplitude are parallel to the interface no matter what the angle of incidence θ_0

$$k_0 \sin\theta_0 = k_2' \sin\theta_2'$$

$\leftarrow$ need two equations to solve for k_2' and θ_2'. The 2nd equation comes from dispersion relation in medium (b)

$$k_0 = \frac{\omega}{c}\sqrt{\mu_a \epsilon_a} = \frac{\omega}{c} n_a$$

planes of constant phase are $\perp$ to $\vec{k}_2'$

dispersion relation

$$k_2^2 = \vec{k}_2 \cdot \vec{k}_2 = (k_2')^2 - (k_2'')^2 + 2i\, \vec{k}_2' \cdot \vec{k}_2'' = \frac{\omega^2}{c^2}\mu_b \epsilon_b$$

$$\vec{k}_2' \cdot \vec{k}_2'' = k_2' k_2'' \cos\theta_2'$$

equate real and imaginary parts

$$(k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2}\mu_b \epsilon_{b1} \qquad \epsilon_b = \epsilon_{b1} + i\epsilon_{b2}$$

real

$$2 k_2' k_2'' \cos\theta_2' = \frac{\omega^2}{c^2}\mu_b \epsilon_{b2}$$

solve

$$(k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$(k_2')^2 = \left(\frac{\frac{\omega^2}{c^2}\mu_b \epsilon_{b2}}{2 k_2' \cos\theta_2'}\right)^2 + \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$(k_2')^4 - \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}(k_2')^2 - \frac{\omega^4}{c^4}\frac{\mu_b^2 \epsilon_{b2}^2}{4\cos^2\theta_2'} = 0$$

quadratic formula

$$(k_2')^2 = \frac{\omega^2 \mu_b \epsilon_{b1}}{c^2\, 2} + \sqrt{\frac{\omega^4 \mu_b^2 \epsilon_{b1}^2}{c^2\, 4} + \frac{\omega^4 \mu_b^2 \epsilon_{b2}^2}{c^2\, 4\cos^2\theta_2'}}$$

$$\boxed{k_2' = \frac{\omega}{c}\sqrt{\mu_b}\left[\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

and

$$(k_2'')^2 = (k_2')^2 - \frac{\omega^2}{c^2}\mu_b \epsilon_{b1}$$

$$\boxed{k_2'' = \frac{\omega}{c}\sqrt{\mu_b}\left[-\frac{1}{2}\epsilon_{b1} + \frac{1}{2}\sqrt{\epsilon_{b1}^2 + \frac{\epsilon_{b2}^2}{\cos^2\theta_2'}}\right]^{1/2}}$$

Note, these reduce to what we had earlier for a plane wave, if we take $\theta_2' = 0$

Both k_2' and k_2'' depend on angle of refraction θ_2'

Finally: $k_2' \sin\theta_2' = \frac{\omega}{c} n_a \sin\theta_0$

$$\Rightarrow \boxed{n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\epsilon_{b2}^2}{\epsilon_{b1}^2\cos^2\theta_2'}}\right]^{1/2} \sin\theta_2'}$$

determines θ_2' in terms of given θ_0

Cases

(1) for a nearly transparent material with $\epsilon_{b2} \ll \epsilon_{b1}$

define $n_b = \sqrt{\mu_b \epsilon_{b1}}$ index of refraction

$$n_a \sin\theta_0 = n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{4\epsilon_{bi}^2 \cos^2\theta_2'}\right]^{1/2}$$

$$\simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{bi}^2 \cos^2\theta_2'}\right]$$

small correction to Snell's law

for $\frac{\epsilon_{b2}}{\epsilon_{bi}} \ll 1$ can solve iteratively

to lowest order: $n_a \sin\theta_0 \simeq n_b \sin\theta_2'$

$$\Rightarrow \cos^2\theta_2' = 1 - \sin^2\theta_2' = 1 - \left(\frac{n_a}{n_b}\sin\theta_0\right)^2$$

so to next order

$$n_a \sin\theta_0 \simeq n_b \sin\theta_2' \left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{bi}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]$$

or $$\sin\theta_2' \simeq \frac{n_a}{n_b}\sin\theta_0 \frac{1}{\left[1 + \frac{\epsilon_{b2}^2}{8\epsilon_{bi}^2\left(1 - \frac{n_a^2}{n_b^2}\sin^2\theta_0\right)}\right]}$$

$$\leq \frac{n_a}{n_b}\sin\theta_0$$

result is that θ_2' is <u>smaller</u> than Snell's law would predict.

(2) for a good conductor, or absorbing region of a dielectric, $\epsilon_{b2} \gg \epsilon_{b1}$

to lowest order

$$n_a \sin\theta_0 = \sqrt{\mu_b \epsilon_{b1}} \left[\frac{1}{2} \frac{\epsilon_{b2}}{\epsilon_{b1} \cos\theta_2'} \right]^{1/2} \sin\theta_2'$$

$$n_a \sin\theta_0 = \sqrt{\frac{\mu_b \epsilon_{b2}}{2}} \frac{\sin\theta_2'}{\sqrt{\cos\theta_2'}}$$

very different from Snell's Law!

Snell's Law only holds if both media are transparent