We will see that this situation in general corresponds to \textit{elliptically polarized wave}. \\

\textbf{General Case}: \( E_1 \) and \( E_2 \) are complex constants \\\nwrite \( E_1 \hat{E}_1 + E_2 \hat{E}_2 = \hat{U} e^{i\phi} \) \\\nwhere \( \phi \) is chosen so that \( \hat{U} \cdot \hat{U} \) is real \\\n- one can always do this since \( \hat{U} \cdot \hat{U} = (E_1^2 + E_2^2) e^{-2i\phi} \) \\\nso \( 2\phi \) is just the phase of the complex \( E_1^2 + E_2^2 \) \\\n
\( \hat{U} \) is a complex vector \( \Rightarrow \hat{U} = \hat{U}_a + i \hat{U}_b \) \\\nwith \( \hat{U}_a \) and \( \hat{U}_b \) real vectors \\\nSince \( \hat{U} \cdot \hat{U} \) is real \( \Rightarrow \hat{U}_a \cdot \hat{U}_b = 0 \) \\\nso \( \hat{U}_a \perp \hat{U}_b \) orthogonal \\\n
let \( \hat{e}_a \) be the unit vector in direction of \( \hat{U}_a \) \\\nso \( \hat{U}_a = \|U_a\| \hat{e}_a \) with \( \|U_a\| = 1 \) \\\nlet \( \hat{e}_b = \hat{\mathbf{m}} \times \hat{e}_a \) so that \( \{\hat{\mathbf{m}}, \hat{e}_a, \hat{e}_b\} \) are a right-handed coordinate system \\\n
Then \( \hat{U}_b = \pm \|U_b\| \hat{e}_b \) where \( \|U_b\| = 1 \) \\\n
since \( \hat{U}_a \perp \hat{U}_b \) and both \( \hat{U}_a \) and \( \hat{U}_b \) are \( \perp \) to \( \hat{\mathbf{m}} \). \\\n
It is \((+\) if \( \hat{U}_b \) is parallel to \( \hat{e}_b \) and \nIt is \((-\) if \( \hat{U}_b \) is anti-parallel to \( \hat{e}_b \).
In this representation we have

\[ \tilde{E}(\tilde{r}, t) = \text{Re} \left\{ \tilde{u} e^{i\Psi} e^{-i(k_1 \tilde{r} - \omega t)} \right\} \]

\[ = e^{-k_2 \hat{\mathbf{r}} \cdot \tilde{r}} \text{Re} \left\{ u_a \hat{e}_a e^{i(k_1 \hat{\mathbf{r}} \cdot \tilde{r} - \omega t + \Phi)} \right\} \]

\[ = e^{-k_2 \hat{\mathbf{r}} \cdot \tilde{r}} \left\{ u_a \hat{e}_a \cos(\Psi + \Phi) \right\} \]

where we write \( \Psi = k_1 \hat{\mathbf{r}} \cdot \tilde{r} - \omega t \)

Let's define \( e^{-k_2 \hat{\mathbf{r}} \cdot \tilde{r}} u_a \rightarrow u_a \]

\( e^{-k_2 \hat{\mathbf{r}} \cdot \tilde{r}} u_b \rightarrow u_b \)

so we don't have to keep writing the constant attenuation factor that is a common factor of all components of \( \tilde{E} \).

Then define \( E_a \) and \( E_b \) as the components of \( \tilde{E} \) in the directions \( \hat{e}_a \) and \( \hat{e}_b \) respectively.

\[ E_a = u_a \cos(\Psi + \Phi) \]

\[ E_b = u_b \sin(\Psi + \Phi) \]

This then gives

\[ \left( \frac{E_a}{u_a} \right)^2 + \left( \frac{E_b}{u_b} \right)^2 = \cos^2(\Psi + \Phi) + \sin^2(\Psi + \Phi) = 1 \]

This is just the equation for an ellipse.
with semi-axes of lengths $U_a$ and $U_b$, oriented in the directions of $\hat{e}_a$ and $\hat{e}_b$.

\[ \begin{align*}
\text{E} & \quad (\text{E}) \quad \text{Ea} \\
\text{Ea} & \quad \text{Eb} \\
\text{Ua} & \quad \text{Ub} \\
\text{0} & \quad \text{0}
\end{align*} \]

\[ \Rightarrow \text{At a fixed position } \text{P, the tip of the vector } \text{E} \text{ will trace out the above ellipse as the time increases by one period of oscillation } 2\pi/\omega. \]

For $(\text{+})$, i.e. $\vec{U}_b = U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse **counterclockwise** as $t$ increases.

For $(\text{-})$, i.e $\vec{U}_b = -U_b \hat{e}_b$, $\vec{E}$ goes around the ellipse **clockwise** as $t$ increases.

Such a wave is said to be **elliptically polarized**.

Special cases

1. $U_a = 0$ or $U_b = 0$
   - the wave is **linearly polarized**
(2) \( U_a = U_b \)

The tip of \( \mathbf{E} \) traces out a circle as \( t \) increases. The wave is circularly polarized.

The (+) case is said to have right handed circular polarization.

The (-) case is said to have left handed circular polarization.

One can define circular polarization basis vectors

\[
\hat{\mathbf{e}}_+ = \frac{\hat{\mathbf{e}}_a + i \hat{\mathbf{e}}_b}{\sqrt{2}}, \quad \hat{\mathbf{e}}_- = \frac{\hat{\mathbf{e}}_a - i \hat{\mathbf{e}}_b}{\sqrt{2}}
\]

with \( \hat{\mathbf{e}}_a \) and \( \hat{\mathbf{e}}_b \) orthogonal.

A wave with complex amplitude \( \hat{E}_w = E \hat{e}_+ \) is right handed circularly polarized.

A wave with complex amplitude \( \hat{E}_w = E \hat{e}_- \) is left handed circularly polarized.

Just as the general case can always be written as a superposition of two orthogonal linearly polarized waves, i.e.

\[
\hat{E}_w = E_1 \hat{e}_1 + E_2 \hat{e}_2
\]
one can also always write the general case as a superposition of a left handed and a right handed circularly polarized wave

$$\mathbf{U} = \mathbf{U}_a + i \mathbf{U}_b = \mathbf{U}_a \hat{e}_a + i \mathbf{U}_b \hat{e}_b$$

$$= \left( \frac{\mathbf{U}_a + \mathbf{U}_b}{\sqrt{2}} \right) \hat{e}_+ + \left( \frac{\mathbf{U}_a - \mathbf{U}_b}{\sqrt{2}} \right) \hat{e}_-$$

(resubstitute in for $\hat{e}_\pm$ and expand, to see that this is so)

⇒ An elliptically polarized wave can be written as a superposition of circularly polarized waves

As a special case of the above (if $\mathbf{U}_a=0$ or $\mathbf{U}_b=0$) a linearly polarized wave can always be written as a superposition of circularly polarized waves.
magnetic field

In the above general formulation we can write \( \vec{H} \) as

\[
\vec{H} = \frac{e}{\omega \mu} \operatorname{Re} \left\{ k \hat{m} \times \vec{U} e^{i \Phi} e^{i (k \cdot \hat{r} - \omega t)} \right\}
\]

\[
= \frac{c |k|}{\omega \mu} \operatorname{Re} \left\{ \hat{m} \times (U_a \hat{e}_a \pm i U_b \hat{e}_b) e^{i (k \cdot \hat{r} - \omega t + \delta + \Phi)} \right\}
\]

\[
= \frac{c |k|}{\omega \mu} \operatorname{Re} \left\{ (U_a \hat{e}_b \mp i U_b \hat{e}_a) e^{i (k \cdot \hat{r} - \omega t + \delta + \Phi)} \right\}
\]

\[
\vec{H} = \frac{c |k|}{\omega \mu} e^{-k_2 \hat{m} \cdot \hat{r}} \left[ U_a \hat{e}_b \cos (\Phi + \delta + \phi) \pm U_b \hat{e}_a \sin (\Phi + \delta + \phi) \right]
\]

we had for the electric field

\[
\vec{E} = e^{-k_2 \hat{m} \cdot \hat{r}} \left[ U_a \hat{e}_a \cos (\Phi + \delta + \phi) \mp U_b \hat{e}_b \sin (\Phi + \delta + \phi) \right]
\]

Consider \( \vec{E} \cdot \vec{H} \). From the above, with \( \hat{e}_a \cdot \hat{e}_b = 0 \), we get

\[
\vec{E} \cdot \vec{H} = e^{-2k_2 \hat{m} \cdot \hat{r}} \operatorname{Re} \left\{ \frac{c |k|}{\omega \mu} U_a U_b (\pm 1) \left[ \sin (\Phi + \delta + \phi) \cos (\Phi + \delta + \phi) \right. \right.

\[
- \cos (\Phi + \delta + \phi) \sin (\Phi + \delta + \phi) \left. \right] \right\}
\]

\[
= e^{-2k_2 \hat{m} \cdot \hat{r}} \frac{c |k|}{\omega \mu} U_a U_b (\pm 1) \sin \delta
\]

where in the last step we used \( \sin A \cos B - \cos A \sin B = \sin (A - B) \)

We see that \( \vec{E} \cdot \vec{H} = 0 \) only when

1) \( \delta = 0 \), i.e., the medium has no absorption

or

2) \( U_a = 0 \) or \( U_b = 0 \), i.e., the wave is \underline{linearly polarized}
Reflection & Transmission of waves at Interfaces

\( \vec{k}_0 \) = incident wave; \( \theta_2 = \) angle of incidence
\( \vec{k}_1 \) = reflected wave; \( \theta_1 = \) angle of reflection
\( \vec{k}_2 \) = the transmitted or "refracted" wave; \( \theta_2 = \) angle of refraction

Let each wave be given by

\[ \vec{F}_n(\vec{r}_0) = \vec{F}_n e^{-i(k_n \cdot \vec{r}_0 - \omega_0 t)} \]

Where \( \vec{F}_n \) can be either \( \vec{E}_n \) or \( \vec{H}_n \) for the electric or magnetic component of the wave.

Boundary condition: tangential component \( \vec{E} \)

must be continuous at \( z=0 \). If \( \vec{E} \) is a vector in \( xy \) plane, and we consider \( \vec{r}_0 = 0 \), then

\[ \hat{\mathbf{x}} \cdot \vec{E}_0 e^{-i\omega_0 t} + \hat{\mathbf{x}} \cdot \vec{E}_1 e^{-i\omega_1 t} = \hat{\mathbf{x}} \cdot \vec{E}_2 e^{-i\omega_2 t} \]

must be true for all time. Can only happen if

\[ \omega_0 = \omega_1 = \omega_2 \equiv \omega \] all frequencies are equal
Now consider the same boundary condition for \( \mathbf{p} \) a position vector in the xy plane at \( z = 0 \). Since we've all agreed we can cancel out the common \( e^{i\mathbf{k} \cdot \mathbf{p}} \) factors to get

\[
\hat{x} \cdot \mathbf{E}_0 e^{i\mathbf{k}_0 \cdot \mathbf{p}} + \hat{x} \cdot \mathbf{E}_1 e^{i\mathbf{k}_1 \cdot \mathbf{p}} = \hat{x} \cdot \mathbf{E}_2 e^{i\mathbf{k}_2 \cdot \mathbf{p}}
\]

this must be true for all \( \mathbf{p} \). Can only happen if the projections of the \( \mathbf{k}_n \) in the xy plane are all equal

\[
\begin{align*}
\hat{x} \cdot \mathbf{k}_0 &= \hat{x} \cdot \mathbf{k}_1 = \hat{x} \cdot \mathbf{k}_2 \\
\hat{y} \cdot \mathbf{k}_0 &= \hat{y} \cdot \mathbf{k}_1 = \hat{y} \cdot \mathbf{k}_2
\end{align*}
\]

only z components \( k_z \) vectors can be different

Choose coordinate system as in diagram so that all \( \mathbf{k}_n \) vectors lie in the xy plane (\( y \) is out of page)

\[
\begin{align*}
\mathbf{E}_0 &= \mathbf{k}_0 \\
\mathbf{E}_1 &= \mathbf{k}_1 \\
\mathbf{E}_2 &= \mathbf{k}_2
\end{align*}
\]

Since \( \mathbf{E}_0 \) is real and positive, therefore one real vectors

\[
\begin{align*}
k_0 &= k_1 \\
\Rightarrow |\mathbf{k}_0| \sin \Theta_0 &= |\mathbf{k}_1| \sin \Theta_1
\end{align*}
\]

Since \( k_0^2 = \frac{w^2}{c^2} n \mathbf{a} \mathbf{a} \) and \( k_1^2 = \frac{w^2}{c^2} n \mathbf{a} \mathbf{a} \)  7.2.1

Then \( |\mathbf{k}_0| = |\mathbf{k}_1| \) so \( \sin \Theta_0 = \sin \Theta_1 \)

\[
\Theta_0 = \Theta_1
\]

Angle of incidence = angle of reflection
If \(\varepsilon_b\) is also real and positive (B is transparent) then \(|k_2|\) is real

\[ k_{ox} = k_{2x} \Rightarrow |k_0| \sin \theta_0 = |k_2| \sin \theta_2 \]

\[ k_2^2 = \frac{\omega^2}{c^2} \varepsilon_b \mu_b \]

\[ \Rightarrow \sqrt{\mu_0 \varepsilon_0} \sin \theta_0 = \sqrt{\mu_0 \varepsilon_0} \sin \theta_2 \]

In terms of index of refraction \( M = \frac{k_e}{\omega} = \frac{\omega}{c} \sqrt{\mu \varepsilon} \)

\[ M = \sqrt{\mu \varepsilon} \]

\[ \Rightarrow m_a \sin \theta_0 = m_b \sin \theta_2 \]

\[
\begin{array}{c|c}
\sin \theta_2 & m_a \\
\sin \theta_0 & m_b \\
\end{array}
\]

Strelo's Law

time for all types of waves, not just EM waves

If \(m_a > m_b\) then \(\theta_2 > \theta_0\)

In this case, when \(\theta_2\) is too large, we will have

\[ \frac{m_a}{m_b} \sin \theta_0 > 1 \]

and there will be no solution for \(\theta_2\)

\[ \Rightarrow \text{no transmitted wave} \]

This is "total internal reflection" - wave does not exit medium B. The critical angle \(\theta_c\) above which one has total internal reflection is given by

\[ \frac{m_a}{m_b} \sin \theta_c = 1 \]

\[ \theta_c = \arcsin \left( \frac{m_b}{m_a} \right) \]
Density

\[ E = 1 + 4\pi\alpha x \]

Since \( n = \sqrt{n_E} \) and \( n \) grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

**Examples:** diamonds sparkle due to total internal reflection. Diamonds have large \( n \) \( \Rightarrow \) small \( \theta \) \( \Rightarrow \) light bounces around inside many times before it can exit.

Can also see total internal reflection when swimming under water.

**More general case:** \( \sqrt{n_E} \) is complex so \( \vec{k}_2 \) is complex

\[ \vec{k}_2 = \vec{k}_2' + i\vec{k}_2'' \]

\[ \vec{k}_2' = |\vec{k}_2'| \]

\[ \vec{k}_2'' = |\vec{k}_2''| \]

real part, imaginary part

**Note:** \( \vec{k}_2' \) and \( \vec{k}_2'' \) need not be in the same direction!

Condiiton \( \vec{k}_{ox} = \vec{k}_{2x} \Rightarrow \begin{cases} \vec{k}_{ox} = \vec{k}_{2x} \Rightarrow \text{equate real and imaginary parts} \end{cases} \]

\[ \vec{k}_o \sin \theta_0 = \vec{k}_2' \sin \theta_2' \]

\[ \theta = \vec{k}_2'' \sin \theta_2'' \]
\[ \Rightarrow \Theta_2'' = 0 \]

\[ k_2'' = k_2'' \hat{z} \]

\[ \text{Attenuation factor for the transmitted wave is } e^{-k_2' z} \]

\[ \rightarrow \text{planes of constant amplitude are parallel to the interface no matter what the angle of incidence } \Theta_0. \]

\[ k_0 \sin \Theta_0 = k_0' \sin \Theta_0' \]

\[ k_0 = \frac{\omega}{c} \sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} \mu_0 \]

\[ \text{need two equations to solve for } k_2' \text{ and } \Theta_2' \]

\[ \text{The 2nd equation comes from dispersion relation in medium (b).} \]

\[ \text{planes of constant phase are } \perp \text{ to } \mathbf{k}_2' \]

\[ -k_2^2 = \mathbf{k}_2 \cdot \mathbf{k}_2 = (k_2')^2 + (k_2'')^2 + 2i k_2' \cdot k_2'' = \frac{\omega^2}{c^2} \mu_b \epsilon_b \]

\[ \mathbf{k}_2' \cdot \mathbf{k}_2'' = k_2' k_2'' \cos \Theta_2' \]

\[ \text{Equate real and imaginary parts} \]

\[ (k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2} \mu_b \epsilon_{b1} \]

\[ \epsilon_b = \epsilon_{b1} + i \epsilon_{b2} \]

\[ \frac{2 k_2' k_2'' \cos \Theta_2'}{\omega^2} = \frac{\omega^2}{c^2} \mu_b \epsilon_{b2} \]

\[ \text{Solve} \]

\[ (k_2')^2 = (k_2'')^2 + \frac{\omega^2}{c^2} \mu_b \epsilon_{b1} \]

\[ (k_2')^2 = \left( \frac{\omega^2}{c^2} \mu_b \epsilon_{b2} \frac{1}{2 k_2' \cos \Theta_2'} \right)^2 + \frac{\omega^2}{c^2} \mu_b \epsilon_{b1} \]
\[(k_2')^4 - \frac{c^2 \mu_0 E_b k_2'}{c^4} (k_2')^2 - \frac{\mu_0^2 E_b^2}{4 c^4} \frac{\mu_0 E_b}{4 \cos^2 \theta_2} = 0\]

Quadratic formula:

\[k_2' = \frac{c^2 \sqrt{\mu_0}}{2} \left[ \frac{1}{2} E_b + \frac{1}{2} \sqrt{E_b^2 + \frac{E_b^2}{\cos^2 \theta_2}} \right]^{1/2} \]

And:

\[k_2'' = (k_2')^2 - \frac{\omega^2}{c^2} \mu_0 E_b \]

\[k_2'' = \frac{c^2 \sqrt{\mu_0}}{2} \left[ -\frac{1}{2} E_b + \frac{1}{2} \sqrt{E_b^2 + \frac{E_b^2}{\cos^2 \theta_2}} \right]^{1/2} \]

Note, these reduce to what we had earlier for a plane wave, if we take \( \theta_2 = 0 \).

Both \( k_2' \) and \( k_2'' \) depend on angle of refraction \( \theta_2 \).

Finally, \( k_2' \sin \theta_2' = \omega n_0 \sin \theta_0 \)

\[
\sin \theta_0 = \sqrt{\frac{\mu_0 E_b}{c^4}} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{E_b^2}{E_b^2 \cos^2 \theta_2}} \right]^{1/2} \sin \theta_2' \]

determines \( \theta_2' \) in terms of given \( \theta_0 \).

Cases:

1) for a nearly transparent material with \( E_b \ll E_b' \)

\[n_0 = \sqrt{\mu_0 E_b} \]

defines \( n_0 = \sqrt{\mu_0 E_b} \) index of refraction
\[
m_a \sin \theta_0 = m_b \sin \theta_2' \left[ 1 + \frac{E_{b2}^2}{4 E_{b1}^2 \cos^2 \theta_2'} \right]^{1/2}
\]

\[
\approx m_b \sin \theta_2' \left[ 1 + \frac{E_{b2}^2}{8 E_{b1}^2 \cos^2 \theta_2'} \right]
\]

Small correction to Snell's law for \( E_{b2} \ll 1 \) can solve iteratively\( \frac{E_{b1}}{E_{b2}} \)

to lowest order:

\[
m_a \sin \theta_2' = m_b \sin \theta_2' \Rightarrow \cos^2 \theta_2' = 1 - \sin^2 \theta_2' = 1 - \left( \frac{m_a \sin \theta_0}{m_b} \right)^2
\]

So to next order,

\[
m_a \sin \theta_2' = m_b \sin \theta_2' \left[ 1 + \frac{E_{b2}^2}{8 E_{b1}^2 \left( 1 - \frac{m_a^2}{m_b^2} \sin^2 \theta_2' \right)} \right]
\]

or \( \sin \theta_2' = \frac{m_a \sin \theta_0}{m_b} \left[ 1 + \frac{E_{b2}^2}{8 E_{b1}^2 \left( 1 - \frac{m_a^2}{m_b^2} \sin^2 \theta_0 \right)} \right] \)

\[
\frac{E_{b1}}{E_{b2}} \geq \frac{m_a}{m_b} \sin \theta_0
\]

result is that \( \theta_2' \) is smaller than Snell's law would predict.
for a good conductor, or absorbing region of a dielectric, $E_b_2 \gg E_b_1$

to lowest order

$$M_a \sin \theta_0 = \sqrt{M_b E_b_1} \left( \frac{1}{2} \frac{E_b_2}{E_b_1 \cos \theta_2} \right)^{1/2} \sin \theta_2$$

$$M_a \sin \theta_0 = \sqrt{\frac{M_b E_b_2}{2}} \frac{\sin \theta_2}{\sqrt{\cos \theta_2}}$$

Snell's law only holds if both media are transparent.

very different from Snell's Law!