

For charge moving with constant velocity along \hat{z}

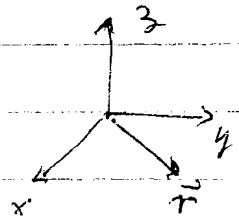
$$\vec{r}_0(t) = vt \hat{z} \quad \vec{v} = \frac{d\vec{r}_0}{dt} = v\hat{z}$$

(in xy plane)

For observer at position \vec{r} , time t , the fields will be determined by the charge at time t' such that

$$t - t' - \frac{|\vec{r} - \vec{r}_0(t')|}{c} = 0$$

$$t - t' - \frac{\sqrt{r^2 + v^2 t'^2}}{c} = 0$$



$$(t - t')^2 = \frac{r^2 + v^2 t'^2}{c^2}$$

$$(1 - v^2/c^2) t'^2 - 2tt' + t^2 - \frac{r^2}{c^2} = 0$$

let $\gamma = (1 - v^2/c^2)^{-1/2}$

$$t'^2 - 2\gamma^2 t t' + \gamma^2 (c^2 t^2 - r^2) = 0$$

$$t' = \gamma^2 t \pm \sqrt{\gamma^4 t^2 - \gamma^2 t^2 + \gamma^2 \frac{r^2}{c^2}}$$

$$= \gamma^2 t \pm \sqrt{\gamma^2 (\gamma^2 t^2 - t^2 + \frac{r^2}{c^2})}$$

$$\gamma^2 - 1 = \frac{1}{1 - v^2/c^2} - 1 = \frac{v^2/c^2}{1 - v^2/c^2} = \gamma^2 \frac{v^2}{c^2}$$

$$= \gamma^2 t \pm \gamma \sqrt{t^2 \gamma^2 \left(\frac{v^2}{c^2} \right) + \frac{r^2}{c^2}}$$

$$t' = \gamma^2 t \pm \frac{\gamma^2}{c} \sqrt{v^2 t^2 + \frac{r^2}{\gamma^2}}$$

consider $t=0$. solution should give $t' < 0$
 $\Rightarrow (-)$ sign is the solution we want

$$t' = \gamma^2 t - \frac{\gamma^2}{c} \sqrt{v^2 t^2 + r^2}$$

$$\phi(\vec{r}, t) = \frac{q}{|\vec{r} - \vec{r}_0(t')| \left[1 - \frac{1}{c} \dot{\vec{M}}(t') \cdot \vec{v} \right]}$$

~~$$\vec{r}_0(t') = \vec{r}_0 + \vec{v} t'$$~~

$$|\vec{r} - \vec{r}_0(t')| = \sqrt{r^2 + v^2 t'^2} = c(t - t') \quad \leftarrow \text{from condition that determines } t'$$

$$(\vec{r} - \vec{r}_0(t')) \cdot \vec{v} = -\vec{r}_0(t') \cdot \vec{v} \quad \text{for } \vec{v} = v \hat{z}$$

$$= -v^2 t' \quad \vec{r} \text{ in } xy \text{ plane}$$

$$\phi(\vec{r}, t) = \frac{q}{c(t - t') \left[1 + \frac{v^2 t'}{c^2(t - t')} \right]}$$

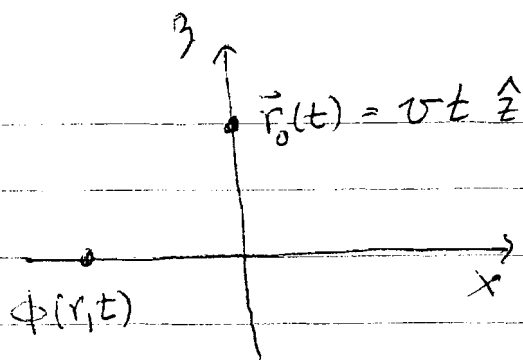
$$= \frac{q}{c(t - t') + \frac{v^2 t'}{c}} = \frac{q}{c \left[t - \left(1 - \frac{v^2}{c^2}\right) t' \right]}$$

$$= \frac{q}{c \left(t - \frac{t'}{\gamma^2} \right)} = \frac{q}{c \frac{1}{\gamma^2} \sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

$$\phi(\vec{r}, t) = \frac{q}{\sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{v}}{c \sqrt{v^2 t^2 + r^2 / \gamma^2}}$$

solutions for
 \vec{r} in xy plane
 when charge passes
 through xy plane
 at $t=0$



at x
potential from charge at $vt \hat{z}$

potential at pt \vec{r} in xy plane
at time t , when charge is at
 $\vec{r}_0 = vt \hat{z}$, looks almost like
static Coulomb potential, which
would be $\frac{q}{\sqrt{r^2 + v^2 t^2}}$

But instead, it is

$$\frac{q}{\sqrt{v^2 t^2 + \left(\frac{r}{\gamma}\right)^2}}$$

looks like the transverse direction has contracted
by a factor γ !

Such considerations led Lorentz to discover
the Lorentz transformation, before Einstein
proposed his theory of special relativity

Radiation from a Localized Oscillating Source

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int d^3r' dt' \frac{\delta(t-t' - \frac{|\vec{r}-\vec{r}'|}{c})}{|\vec{r}-\vec{r}'|} \vec{j}(\vec{r}', t')$$

For pure harmonic oscillation in current

$$\vec{j}(\vec{r}, t) = \text{Re} \left\{ \vec{j}_\omega(\vec{r}) e^{-i\omega t} \right\}$$

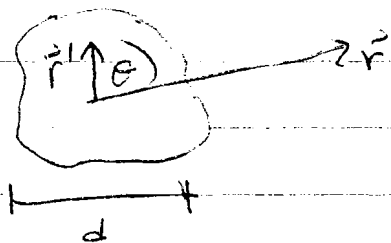
$$\Rightarrow \vec{A}(\vec{r}, t) = \text{Re} \left\{ \vec{A}_\omega(\vec{r}) e^{-i\omega t} \right\}$$

$$\Rightarrow \vec{A}_\omega(\vec{r}) e^{-i\omega t} = \frac{1}{c} \int d^3r' \vec{j}_\omega(\vec{r}') e^{-i\omega t} \frac{e^{i\omega |\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

doing $\int dt'$ by using the δ -function

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c} \int d^3r' \vec{j}_\omega(\vec{r}') \frac{e^{i\omega |\vec{r}-\vec{r}'|/c}}{|\vec{r}-\vec{r}'|}$$

Assume source is localized, i.e. $\vec{j}_\omega(\vec{r}) \approx 0$ for $|\vec{r}| > d$



Approx ①

for $r \gg d$, far from sources

$$\begin{aligned} |\vec{r}-\vec{r}'| &= \sqrt{r^2 + r'^2 - 2rr' \cos \theta} \\ &= r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - \frac{2r'}{r} \cos \theta} \\ &\approx r \left(1 - \frac{r'}{r} \cos \theta\right) \\ &= r - \vec{r}' \cdot \hat{r} + o\left(\left(\frac{r'}{r}\right)^2\right) \\ &\quad \hat{r} \equiv \frac{\vec{r}}{r} \end{aligned}$$

$$\vec{A}_\omega(\vec{r}) = \frac{1}{c} \int d^3r' \frac{\vec{j}_\omega(\vec{r}') e^{-ik(r-\vec{r}' \cdot \hat{r})}}{r-\vec{r}' \cdot \hat{r}} \quad \text{where } k \equiv \frac{\omega}{c}$$

$$= \frac{e^{ikr}}{cr} \int d^3r' \frac{\vec{j}_\omega(\vec{r}') e^{-ik\vec{r}' \cdot \hat{r}}}{1 - \frac{\hat{r} \cdot \vec{r}'}{r}}$$

$$\approx \frac{e^{ikr}}{cr} \int d^3r' \vec{j}_\omega(\vec{r}') e^{-ik\hat{r} \cdot \vec{r}'} \left(1 + \frac{\hat{r} \cdot \vec{r}'}{r}\right)$$

when combine with the $e^{-i\omega t}$ piece, this gives outgoing spherical wave $\frac{e^{i(kr-\omega t)}}{r}$

oscillating charge radiates outgoing spherical electromagnetic waves

the $\int d^3r' \vec{j}_\omega(\vec{r}') \dots$ term will determine the angular dependence of the radiation.

Approx ② $\lambda \gg d$ long wave length approx

$$\text{or } kd \ll 1 \rightarrow \frac{\omega}{c} d \ll 1 \text{ or } \frac{d}{\tau} \ll c$$

where τ is period of oscillation.

Since $\frac{d}{\tau}$ is max speed of the oscillating charges $\rightarrow \lambda \gg d$ is a non-relativistic approximation

$$kd \ll 1 \Rightarrow e^{-ik\hat{r} \cdot \vec{r}'} \approx 1 - ik\hat{r} \cdot \vec{r}' + \text{higher orders}$$

$$\begin{aligned} \vec{A}_\omega(\vec{r}) &= \frac{e}{cr} \int d^3r' \vec{j}_\omega(\vec{r}') (1 - ik\hat{r} \cdot \vec{r}') \left(1 + \frac{\hat{r} \cdot \vec{r}'}{r}\right) \\ &= \frac{e^{ikr}}{cr} \int d^3r' \vec{j}_\omega(\vec{r}') \left[1 + \hat{r} \cdot \vec{r}' \left(\frac{1}{r} - ik\right) \right] \\ &\quad + \text{higher order in } \frac{d}{r} \text{ or } kd \end{aligned}$$

$$\vec{A}_\omega(\vec{r}) = \frac{e^{ikr}}{r} \left[\vec{I}_1 + \left(\frac{1}{r} - ik\right) \vec{I}_2 \right]$$

$$\text{where } \vec{I}_1 \equiv \frac{1}{c} \int d^3r' \vec{j}_\omega(\vec{r}')$$

$$\vec{I}_2 \equiv \frac{1}{c} \int d^3r' \hat{r} \cdot \vec{r}' \vec{j}_\omega(\vec{r}')$$

Consider first \vec{I}_1 its component (\vec{I}_1 vanishes in statics)

$$\int d^3r j_i(\vec{r}) = - \int d^3r r_i \vec{\nabla} \cdot \vec{j} \quad \text{integration by parts}$$

$$= \int d^3r r_i \frac{\partial f}{\partial t} \quad \text{as } \vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

$$\int d^3r j_i(\vec{r}) = -i\omega \int d^3r r_i f_\omega(\vec{r})$$

$$\Rightarrow \vec{I}_1 = -\frac{i\omega}{c} \int d^3r \vec{r} f_\omega(\vec{r}) = -\frac{i\omega \vec{P}_\omega}{c} \quad \begin{array}{l} \uparrow \\ \text{electric} \\ \text{dipole moment} \end{array}$$

Electric dipole approximation from \vec{I}_1

$$\vec{A}_{E1}(\vec{r}) = \frac{e^{i\vec{k}\cdot\vec{r}}}{r} (-i\omega\vec{p}_0) = -i\vec{p}_0 \frac{k e^{i\vec{k}\cdot\vec{r}}}{r} \quad \omega = ck$$

Consider \vec{I}_2

$$\vec{I}_2 = \frac{1}{c} \int d^3r' \hat{r} \cdot \vec{r}' \vec{j}_\omega(\vec{r}') = \frac{1}{c} \hat{r} \cdot \int d^3r' \vec{r}' \vec{j}_\omega(\vec{r}') \quad \text{tensor}$$

we saw this tensor earlier when we did the magnetic dipole approx, and when we derived the macroscopic Maxwell equations

$$\begin{aligned} \int d^3r' \vec{r}' \vec{j}_\omega(\vec{r}') &= - \int d^3r' \vec{j}_\omega(\vec{r}') \vec{r}' - \int d^3r' \vec{r}' \hat{r} (\vec{\nabla}' \cdot \vec{j}_\omega(\vec{r}')) \\ &= \frac{1}{2} \int d^3r' [\vec{r}' \vec{j}_\omega - \vec{j}_\omega \vec{r}'] - \frac{1}{2} \int d^3r' \hat{r} \vec{r}' \vec{r}' \rho_\omega \end{aligned}$$

using $\vec{\nabla}' \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$

$$\begin{aligned} \vec{I}_2 &= \frac{1}{2c} \int d^3r' [(\hat{r} \cdot \vec{r}') \vec{j}_\omega - (\hat{r} \cdot \vec{j}_\omega) \vec{r}'] - \frac{1}{2} \frac{\omega}{c} \hat{r} \cdot \int d^3r' (\vec{r}' \vec{r}') \rho_\omega(\vec{r}') \\ &= -\frac{1}{2c} \int d^3r' [\hat{r} \times (\vec{r}' \times \vec{j}_\omega)] - \frac{1}{2} \frac{\omega}{c} \hat{r} \cdot \int d^3r' (\vec{r}' \vec{r}') \rho_\omega(\vec{r}') \\ &= -\hat{r} \times \vec{m}_\omega - \frac{1}{2} \frac{i\omega}{3c} \hat{r} \cdot \vec{Q}_\omega \end{aligned}$$

where $\vec{m}_\omega = \frac{1}{2c} \int d^3r' \vec{r}' \times \vec{j}_\omega(\vec{r}')$ is magnetic dipole moment

$\vec{Q}_\omega = \int d^3r' 3\vec{r}' \vec{r}' \rho_\omega(\vec{r}')$ looks almost like electric quadrupole tensor

to make it look like the proper quadrupole moment

$$\vec{Q}_\omega = \int d^3r (3\vec{r}'\vec{r}' - r'^2 \vec{I}) \rho_\omega(\vec{r}')$$

we can write

$$\vec{Q}'_\omega = \vec{Q}_\omega + \vec{I} \int d^3r' r'^2 \rho_\omega(\vec{r}')$$

↑ identity matrix $I_{ij} = \delta_{ij}$

$$\vec{I}_2 = -\hat{r} \times \vec{m}_\omega - \frac{1}{2} \frac{i\omega}{3c} \hat{r} \cdot \vec{Q}_\omega - \frac{i\omega}{6c} \hat{r} C(\omega)$$

$$\text{where } C_\omega \equiv \int d^3r' r'^2 \rho_\omega(\vec{r}')$$

is a scalar

Magnetic dipole approximation from \vec{I}_2

$$\vec{A}_{M1}(\vec{r}) = \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik \right) (-\hat{r} \times \vec{m}_\omega)$$

Electric quadrupole approximation from \vec{I}_2

$$\vec{A}_{E2}(\vec{r}) = \frac{e^{ikr}}{r} \left(\frac{1}{r} - ik \right) \left(-\frac{i\omega}{6c} \hat{r} \cdot \vec{Q}_\omega \right)$$

The last piece $\frac{e^{ikr}}{r} \left(\frac{1}{r} - ik \right) \left(-\frac{i\omega}{6c} \hat{r} C_\omega \right)$

can always be ignored - it is a radial function

and so its curl always vanishes \rightarrow gives

no contribution to \vec{B} . Similarly, since $-\frac{i\omega}{c} \vec{E}_\omega = \text{curl } \vec{B}_\omega$

by Ampere's law, this term will give no contribution to \vec{E} .

↑
holds away
from source
where $r \neq 0$.

So with these two approximations ① and ②

$$\vec{A}_\omega(\vec{r}) = \vec{A}_{E1}(\vec{r}) + \vec{A}_{M1}(\vec{r}) + \vec{A}_{E2}(\vec{r})$$

keeping higher order terms would give magnetic quadrupole, electric octopole etc.

Compare strengths of the terms above

Approx ③ Radiation zone: for far sources,
 $(r \gg \lambda)$ $\frac{1}{r} \ll k$ so $(\frac{1}{r} - ik) \approx -ik$ in \vec{A}_{M1} and \vec{A}_{E2}
 only keep terms of $O(\frac{1}{r})$

electric dipole $\vec{p}_\omega \sim qd$ $\vec{A}_{E1} \sim qkd$

magnetic dipole $\vec{m}_\omega \sim \frac{v}{c} qd$ $\vec{A}_{M1} \sim qkd \left(\frac{v}{c}\right)$

use $v \sim \frac{d}{c} \sim d\omega \sim dk c \Rightarrow \vec{A}_{M1} \sim q(kd)^2$

electric quadrupole $\vec{Q}_\omega \sim qd^2$ $\vec{A}_{E2} \sim qd^2 k \frac{\omega}{c} \sim q(kd)^2$

Since Approx ② assumed $kd \approx \frac{v}{c} \ll 1$

above is expansion in powers of kd

leading term is electric dipole

next order are

[magnetic dipole
 electric quadrupole]

$$\frac{A_{M1}}{A_{E1}} \sim \frac{A_{E2}}{A_{E1}} \sim kd$$

next order terms are smaller than A_{E1} by factor $(kd)^2$
 etc.