point-charge Biot-Savart law, the actually remain true for all magnetostatic situations

So far

electrostatics
$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 4\pi k_1 \rho & \text{Gauss} \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$$

magnetostatics
$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{j} & \text{Ampere} \end{cases}$$

~~current~~ charge conservation $\quad \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$

Time Dependent situations

Faraday's law of induction $\quad \vec{\nabla}\times\vec{E} \neq 0$!

$$\oint_C d\vec{l}\cdot\vec{E} = -k_3 \frac{\partial}{\partial t}\int_S da\, \hat{n}\cdot\vec{B} \qquad \mathcal{E}_{mf} = -\frac{d\Phi}{dt}$$

emf around loop mag flux

voltage around closed loop ~ − time rate of change of magnetic flux through loop

$$\rightarrow \boxed{\vec{\nabla}\times\vec{E} = -k_3 \frac{\partial \vec{B}}{\partial t}} \qquad k_3 \text{ is universal constant}$$

Maxwell correction to Ampere's law

in our derivation of $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{j}$
we used $\vec{\nabla}\cdot\vec{j} = 0$. This is only true for magnetostatics – it is NOT true in general

Alternatively, since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$ always, if Amperes law was true, we would necessarily conclude that $\vec{\nabla}\cdot\vec{j} = 0$. But $\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t} \neq 0$ in general

Proposed correction : $\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \vec{W}$
where $\vec{W}$ must be such that charge conservation holds.

Now

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0 = 4\pi k_5\, \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\vec{W}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{W} = -4\pi k_5 \vec{\nabla}\cdot\vec{J} = 4\pi k_5 \frac{\partial\rho}{\partial t} \quad \text{by charge conserv}$$

$$= \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} \quad \text{by Gauss Law}$$

$$\Rightarrow \vec{W} = \frac{k_5}{k_1}\frac{\partial \vec{E}}{\partial t}$$

~~So~~ corrected Amperes law is

$$\boxed{\vec{\nabla}\times\vec{B} = 4\pi k_5 \vec{J} + \frac{k_5}{k_1}\frac{\partial\vec{E}}{\partial t}}$$

$\Rightarrow$ EM waves

Now consider $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B}$

$$= -\nabla^2\vec{B} \quad \text{as } \vec{\nabla}\cdot\vec{B} = 0$$

If there are no sources ($\rho = 0$, $\vec{J} = 0$) then

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} = \frac{k_5}{k_1}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}$$

$$= -\frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2} \quad \text{by Faraday}$$

$$\nabla^2 B = \frac{k_5 k_3}{k_1}\frac{\partial^2 B}{\partial t^2} \quad \text{this is the wave equation}$$

$\Rightarrow \frac{k_5 k_3}{k_1}$ has units of $(\text{velocity})^{-2}$

Since we know that the above wave equation describes electromagnetic waves, ie light, then

$$\frac{k_5 k_3}{k_1} = \frac{1}{c^2}$$

we already had $\quad \dfrac{k_4 k_5}{k_1} = \dfrac{1}{c^2}$

$$\Rightarrow k_3 = k_4$$

$\Rightarrow$ k_1 and k_4 are arbitrary - they can be chosen to be anything by adjusting the units of q and B. k_3 and k_5 are then fixed by $\dfrac{k_4 k_5}{k_1} = \dfrac{1}{c^2}$, $k_3 = k_4$

Popular systems of E + M units

	k_1	$k_3 = k_4$	k_5	
MKS or SI	$\frac{1}{4\pi\varepsilon_0}$	1	$\frac{\mu_0}{4\pi}$	$\left(\varepsilon_0\mu_0 = \frac{1}{c^2}\right)$
Gaussian or CGS	1	$\frac{1}{c}$	$\frac{1}{c}$	
Rationalized Gaussian	$\frac{1}{4\pi}$	$\frac{1}{c}$	$\frac{1}{4\pi c}$	

In MKS, charges are measured in "Coulombs"
current measured in "amps"
magnetic field measured in "tesla" = "weber/m^2"

In CGS, charges are measured in "statcoulombs"
current measured in "statamperes"
magnetic field measured in "gauss"

1 tesla $= 10^4$ gauss

We will use the CGS or Gaussian units

1) $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

2) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

3) $\vec{\nabla}\cdot\vec{B} = 0$

4) $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

Maxwell's Eqn's !!

$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$

Lorentz force

Physical content of Maxwell's eqns.

1) Gauss Law for electric field — charge is source of $\vec{E}$ field. Field lines can begin and end at point charges

2) Faraday's Law of induction — time varying magnetic flux produces circulating $\vec{E}$-field

3) Gauss Law for magnetic fields — no magnetic monopoles. Magnetic field lines are continuous, they either close upon themselves or go off to infinity, they cannot begin nor end at any point.

4) Amperes Law + Maxwell's correction — electric current is a source for circulating $\vec{B}$-field; so is a time varying $\vec{E}$-field. Maxwell's correction is necessary to have charge conservation and to give electromagnetic waves.

Note:

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = 0$ as div of curl always vanishes

then (2) $\Rightarrow -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0$

So if $\vec{\nabla}\cdot\vec{B} = 0$ at $t=0$, Equ (2) requires that $\vec{\nabla}\cdot\vec{B}$ remains zero for all time

Similarly

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$

then (4) $\Rightarrow \frac{4\pi}{c}\vec{\nabla}\cdot\vec{j} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = 0$

use charge continuity $\vec{\nabla}\cdot\vec{j} = -\frac{\partial \rho}{\partial t}$ to get

$$-\frac{4\pi}{c}\frac{\partial \rho}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = 0$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{E} - 4\pi\rho\right) = 0$$

So if $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ at $t=0$, Equ (4) requires that $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ for all time

Thus Equs (1) and (3) can be viewed as "initial conditions". If they are true for any particular moment in time, then Equs (2) and (4) ensure that the remain true for all time.

Equs (2) and (3) are also refered to as the homogeneous Maxwell Equ's — they involve only the fields $\vec{E}$ + $\vec{B}$ and not the sources ρ and $\vec{j}$. Eqs (1) and (4) are refered to as the inhomogeneous Maxwell's equs — they involve the sources ρ and $\vec{j}$

Electromagnetic Potentials & Gauge Invariance

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi\vec{j}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Consider first <u>statics</u>.

Electrostatics

$$\vec{\nabla}\times\vec{E} = 0 \quad \text{since } \frac{\partial\vec{B}}{\partial t} = 0 \text{ for statics}$$

From vector calculus we know that if the curl of a vector is ~~also~~ everywhere zero, then we can alway write that vector field as the gradient of some scalar function ϕ

$$\vec{E} = -\vec{\nabla}\phi \Rightarrow \vec{\nabla}\times E = -\vec{\nabla}\times(\vec{\nabla}\phi) = 0$$

ϕ is <u>electrostatic potential</u>

Gauss Law becomes

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot(\vec{\nabla}\phi) = -\nabla^2\phi = 4\pi\rho$$

$$\boxed{\nabla^2\phi = -4\pi\rho} \quad \text{Poisson's Equation}$$

In regions where $\rho = 0$, we have

$$\nabla^2\phi = 0 \qquad \text{Laplace's Equation}$$

In our discussion of Coulomb's Law we saw that the electric field from a distribution of localized charges was

$$\vec{E}(\vec{r}) = \int d^3r' \, \rho(\vec{r}\,') \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$= -\vec{\nabla}\left[\int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right] = -\vec{\nabla}\phi$$

We therefore see that the solution to Poisson's equ for a localized charge distribution ρ (with $\vec{E}=0$ as $\vec{r}\to\infty$) is

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

We will soon spend a fair amount of time learning new ways to solve $\nabla^2\phi=\rho$, both for arbitrary ρ where we want an approx to the above integral (multipole expansion), and for cases where ϕ or $\vec{\nabla}\phi$ are predetermined on the surfaces of specified regions of space, such as conducting surfaces (boundary value problems).

Magnetostatics

$$\vec{\nabla}\cdot\vec{B} = 0$$

From vector calculus we know that if the divergence of a vector field vanishes everywhere, then it can always be written as the curl of anothe vector field $\vec{A}$

$$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$$

$\vec{A}$ is the <u>magnetic vector potential</u>

This remains true <u>in general</u> - not just in magneto<u>statics</u>

Ampere's Law becomes

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} \qquad \left(\text{in magnetostatics } \frac{\partial\vec{E}}{\partial t} = 0\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \frac{4\pi}{c}\vec{j}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \frac{4\pi}{c}\vec{j}$$

<u>magnetostatic gauge inverience</u>

There are many possible vector potentials $\vec{A}$ that result in the same $\vec{B}$. If $\vec{A}$ is such that $\vec{\nabla}\times\vec{A} = \vec{B}$, then $\vec{A}' = \vec{A} + \vec{\nabla}\chi$ also gives $\vec{\nabla}\times\vec{A}' = \vec{B}$, since $\vec{\nabla}\times\vec{\nabla}\chi = 0$ for <u>any</u> scalar function $\chi(\vec{r})$.

Therefore we can always choose to represent $\vec{B}$ by a vector potential $\vec{A}$ such that $\vec{\nabla}\cdot\vec{A}=0$.

<u>proof</u>: Suppose we had $\vec{B}=\vec{\nabla}\times\vec{A}$ for some $\vec{A}$ with $\vec{\nabla}\cdot\vec{A}=D(\vec{r})\neq 0$. Construct an $\vec{A}'=\vec{A}+\vec{\nabla}\chi$ with χ chosen as follows:

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi = 0 \quad\Rightarrow\quad \nabla^2\chi = -\vec{\nabla}\cdot\vec{A} = D$$

Solve for χ, for exaple

$$\chi(\vec{r}) = \int d^3r' \frac{D(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}$$

we thus have constucted an $\vec{A}'$ such that $\vec{\nabla}\times\vec{A}'=\vec{B}$ <u>and</u> $\vec{\nabla}\cdot\vec{A}'=0$.

This freedom to choose various $\vec{A}$'s that give the same $\vec{B}$ is called <u>gauge invarience</u>. Imposing a particular additional constraint on $\vec{A}$ that removes this freedom is called <u>fixing the gauge</u>. The choice $\vec{\nabla}\cdot\vec{A}=0$ is usually known as the <u>Coulomb gauge</u> (or sometimes the <u>Landau gauge</u>). Going from $\vec{A}$ to $\vec{A}'=\vec{A}+\vec{\nabla}\chi$ is called making a <u>gauge transformation</u>.

"Working in the Coulomb gauge" with $\vec{\nabla}\cdot\vec{A}=0$, Amperes Law becomes

$$\boxed{\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{j}} \qquad \text{Poisson's Equ.}$$

For a localized current density

$$\vec{A}(\vec{r}) = \frac{1}{c}\int d^3r' \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

Back to dynamics

$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ remains true

But now instead of $\vec{\nabla}\times\vec{E} = 0$ we have

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) = 0$$

$$\Rightarrow \quad \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0$$

$\Rightarrow$ there exists a scalar potential ϕ such that

$$\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi \quad \text{or} \quad \boxed{\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}}$$

Gauss's law for electric field now becomes

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = 4\pi\rho$$

$$\boxed{\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho}$$ Gauss law in terms of electromagnetic potentials

Ampere's Law becomes

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\phi - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}$$

$$\boxed{-\nabla^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\phi + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)}$$

or

$$\boxed{-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}$$

<u>gauge invarience</u>

As before, we can always construct $\vec{A}' = \vec{A} + \vec{\nabla}\chi$, for any scalor function χ, that gives the same $\vec{B}$. But since $\vec{A}$ now also enters expression for $\vec{E}$, we need to make sure that if we change $\vec{A}$ to $\vec{A}'$, we must make some corresponding change ϕ to ϕ' so that $\vec{E}$ does not change.

$$\left[\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\chi \\ \phi' &= \phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \end{aligned}\right] \quad \text{gauge transformation}$$

For any scalor χ, the above $\vec{A}'$ and ϕ' give the same values of $\vec{E}$ and $\vec{B}$ as $\vec{A}$ and ϕ.

<u>Proof:</u>

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \vec{\nabla}\times\vec{\nabla}\chi = \vec{\nabla}\times\vec{A} = \vec{B}$$

$$\left(-\vec{\nabla}\phi' - \frac{1}{c}\frac{\partial\vec{A}'}{\partial t}\right) = -\vec{\nabla}\phi + \frac{1}{c}\vec{\nabla}\frac{\partial\chi}{\partial t} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\chi$$

$$= \left(-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = \vec{E}$$

As before, we can <u>fix the gauge</u> by imposing some additional constraint on $\vec{A}$ and ϕ. There are <u>two</u> popular choices:

1) <u>Lorentz Gauge</u>

<u>gauge constraint</u>: require $\frac{1}{c}\frac{\partial\phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0$

Then Gauss' Law becomes

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -4\pi\rho$$

$$\Rightarrow \nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\phi}{\partial t}\right) = -4\pi\rho$$

$$\boxed{\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho}$$

Ampere's Law becomes

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{4\pi}{c}\vec{j} - \vec{\nabla}\underbrace{\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right)}_{=0}$$

$$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{j}}$$

The combination $-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \equiv \Box^2$ is the <u>wave equation</u> operator.

In Lorentz gauge, $\vec{A}$ and ϕ satisfy the inhomogeneous wave equations:

$$\boxed{\begin{aligned}\Box^2\vec{A} &= \frac{4\pi}{c}\vec{j}\\ \Box^2\phi &= 4\pi\rho\end{aligned}}$$

when $\vec{j}=0$, $\rho=0$ electromagnetic waves are solution!