Spherical Coordinates

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

$$\phi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

$$r^2\nabla^2\phi = \Theta\Phi\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2} = 0$$

$$\frac{r^2\sin^2\theta\,\nabla^2\phi}{\phi} = \underbrace{\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\substack{\text{depends only on } r \text{ and } \theta \\ = -\text{const}}} + \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\substack{\text{depends only on } \varphi \\ = \text{const}}} = 0$$

take $\displaystyle \frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2} = -m^2$

$$\Rightarrow \boxed{\Phi = e^{\pm i m\varphi}}$$

m integer for 2π periodicity in φ

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = m^2$$

$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\substack{\text{depends only on } r \\ = \text{const}}} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\substack{\text{depends only on } \theta \\ = -\text{const}}} = 0$$

call the const $\ell(\ell+1)$

For R

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \ell(\ell+1) = 0$$

Solutions are of the form $\boxed{R(r) = a_\ell r^\ell + b_\ell r^{-(\ell+1)}}$
substitute in to verify

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d}{dr}\left(r^2\left(\ell a_\ell r^{\ell-1} - (\ell+1)b_\ell r^{-\ell-2}\right)\right)$$

$$= \frac{d}{dr}\left(\ell a_\ell r^{\ell+1} - (\ell+1)b_\ell r^{-\ell}\right)$$

$$= \ell(\ell+1)a_\ell r^\ell + \ell(\ell+1)b_\ell r^{-(\ell+1)} = \ell(\ell+1)R$$

For Θ:
$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\ell(\ell+1)$$

let $x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$d\theta = -\dfrac{dx}{\sin\theta}$

$0 \leq \theta \leq \pi$

solutions for $-1 \leq x \leq 1$
correspond to $\ell \geq 0$ integers

above becomes

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

Called generalized Legendre Equation - solutions are called the associated Legendre functions.

ordinary Legendre polynomials are solutions for $m=0$

For the special case $m=0$, ie the solution has azimuthal symmetry and Φ does not depend on the angle φ (ie rotational symmetry about $\hat{z}$ axis),

We want the solutions to

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \ell(\ell+1)\,\Theta = 0$$

The solutions are known as the Legendre polynomials, $P_\ell(x)$.

They are given, for ℓ integer, by

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \qquad \text{Rodriguez's formula}$$

The lowest ℓ polynomials are

$$P_0(x) = 1 \qquad\qquad P_2(x) = \tfrac{1}{2}(3x^2-1)$$
$$P_1(x) = x \qquad\qquad P_3(x) = \tfrac{1}{2}(5x^3-3x)$$

In general, $P_\ell(x)$ is a polynomial of order ℓ with only even powers if ℓ is even, and only odd powers if ℓ is odd. $\Rightarrow P_\ell(x) \begin{cases} \text{even in } x \text{ for } \ell \text{ even} \\ \text{odd in } x \text{ for } \ell \text{ odd} \end{cases}$

$P_\ell(x)$ is normalized so that $P_\ell(1) = 1$

Note: Legendre polynomials are only for integer $\ell \geq 0$.
What about solutions for non integer ℓ?
The $P_\ell(x)$ give one solution for each integer ℓ.
But $P_\ell(x)$ are defined by a 2nd order differential equation - shouldn't there be a 2nd independent solution for each ℓ?
It turns out that these "2nd" solutions, as well as solutions for non integer ℓ, all blow up at either $x=-1$ or $x=1$, ie at $\theta=0$ or $\theta=\pi$.
They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval $-1 \leq x \leq 1$.

$$\int_{-1}^{1} dx\, P_\ell(x) P_m(x) = \int_0^\pi d\theta \sin\theta\, P_\ell(\cos\theta)\, P_m(\cos\theta) = \begin{cases} 0 & \ell \neq m \\ \dfrac{2}{2\ell+1} & \ell = m \end{cases}$$

$\Rightarrow$ we can expand any function $f(\theta)$, $0 \leq \theta \leq \pi$, as a linear combination of the $P_\ell(\cos\theta)$.
This is the reason they are useful for solving problems of Laplace's eqn with spherical boundary surfaces.

For $m \neq 0$, the solutions to (see Jackson 3.5)

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]\Theta = 0$$

are the associated Legendre functions $P_\ell^m(x)$.
For $P_\ell^m(x)$ to be finite in interval $-1 \leq x \leq 1$ one again finds that ℓ must be integer $\ell \geq 0$, and integer m must satisfy $|m| \leq \ell$, ie $m = -\ell, -\ell+1, \ldots, 0, \ldots \ell-1, \ell$.
For each ℓ and m there is only one such non divergent solution.

It is typical to combine the solutions $P_\ell^m(\cos\theta)$ to the θ-part of the equation with the $\Phi_m(\varphi) = e^{im\varphi}$ solutions to the φ-part of the equation to define the spherical harmonics

$$Y_{\ell m}(\theta,\varphi) \equiv \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi}$$

The $Y_{\ell m}$ are orthogonal

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

and are a complete set of basis functions for expanding any function $f(\theta,\varphi)$ defined on the surface of a sphere.

Exaples with azimuthal symmetry $m=0$

General solution to $\nabla^2\phi=0$ can be written in form

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right] P_\ell(\cos\theta)$$

determine the A_ℓ and B_ℓ from the boundary conditions of the particular problem.

① Suppose one is given $\phi(R,\theta) = \phi_0(\theta)$ on surface of sphere of radius R.

To find solution of $\nabla^2\phi=0$ inside sphere

ϕ should not diverge at origin $\Rightarrow B_\ell = 0$ for all ℓ

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Rightarrow \phi(R,\theta) = \phi_0(R) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

$$\Rightarrow \int_0^\pi d\theta \sin\theta \, \phi_0(\theta) P_m(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi d\theta \sin\theta \, P_\ell(\cos\theta) P_m(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} A_\ell R^\ell \left(\frac{2}{2\ell+1}\right) \delta_{\ell m}$$

$$= A_m R^m \frac{2}{2m+1}$$

$$\boxed{A_m = \frac{2m+1}{2R^m} \int_0^\pi d\theta \sin\theta \, \phi_0(\theta) P_m(\cos\theta)}$$ gives solution

To find solution of $\nabla^2 \phi = 0$ <u>outside</u> sphere

if require $\phi \to 0$ as $r \to 0$, then $A_\ell = 0$ for all ℓ

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\phi(R,\theta) = \phi_0(\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

gives solution $$\boxed{B_m = \frac{2m+1}{2} R^{m+1} \int_0^\pi d\theta \sin\theta \, \phi_0(\theta) P_m(\cos\theta)}$$

$$\boxed{B_m = A_m R^{2m+1}}$$

② Suppose one is given surface charge density $\sigma(\theta)$ fixed on surface of sphere of radius R. What is ϕ inside and outside?

From previous example

$$\phi(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r < R \\ \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) & r > R \end{cases}$$

boundary conditions at $r = R$ on surface

(i) ϕ continuous

$$\Rightarrow \sum_{\ell=0}^{\infty} \left[A_\ell R^\ell - \frac{B_\ell}{R^{\ell+1}} \right] P_\ell(\cos\theta) = 0$$

If an expansion in Legendre polynomials vanishes for all θ, then each coefficient in the expansion must vanish

$$\Rightarrow \quad A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \quad \Rightarrow \quad \boxed{B_\ell = A_\ell R^{2\ell+1}}$$

(ii) jump in electric field at σ

$$-\left.\frac{\partial \phi^{out}}{\partial r}\right|_{r=R} + \left.\frac{\partial \phi^{in}}{\partial r}\right|_{r=R} = 4\pi\sigma$$

$$\Rightarrow \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)B_\ell}{R^{\ell+2}} + \ell A_\ell R^{\ell-1}\right]P_\ell(\cos\theta) = 4\pi\sigma$$

$$\Rightarrow \sum_{\ell=0}^{\infty}\left[\frac{(\ell+1)A_\ell R^{2\ell+1}}{R^{\ell+2}} + \ell A_\ell R^{\ell-1}\right]P_\ell(\cos\theta)$$

$$\Rightarrow \sum_{\ell=0}^{\infty}(2\ell+1)R^{\ell-1}A_\ell P_\ell(\cos\theta) = 4\pi\sigma$$

$$(2m+1)R^{m-1}A_m\left(\frac{2}{2m+1}\right) = 4\pi\int_0^{\pi} d\theta \sin\theta \, \sigma(\theta) P_m(\cos\theta)$$

$$\boxed{A_m = \frac{4\pi}{2R^{m-1}}\int_0^{\pi} d\theta \sin\theta \, \sigma(\theta) P_m(\cos\theta)}$$