Suppose $\sigma(\theta) = k \cos \theta$. What is $\phi$?

Note $\sigma(\theta) = k P_1(\cos \theta)$

hence only $A_1 \neq 0$ by orthogonality of $P_0(\cos \theta)$

$$A_1 = \frac{4\pi k}{2} \int_0^\pi \sin \theta P_1(\cos \theta) P_1(\cos \theta)$$

$$= \frac{4\pi k}{2} \left( \frac{2}{2+1} \right) = \frac{4\pi k}{3}$$

$$\Rightarrow \phi(r, \theta) = \begin{cases} \frac{4\pi k}{3} r \cos \theta & r < R \\ \frac{4\pi k}{3} \frac{R^3}{r^2} \cos \theta & r > R \end{cases}$$

we will see that potential outside the sphere is that of an ideal dipole with dipole moment $p = \frac{4\pi k R^3}{3}$

Inside the sphere, the potential $\phi = \frac{4\pi k}{3} z$

where $z = r \cos \theta$. The electric field inside the sphere is therefore the constant

$$\vec{E} = -\nabla \phi = -\frac{4\pi k}{3} \hat{z}$$
outside the sphere the field is

\[ E = -\nabla \phi = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} \]

\[ = \frac{8\pi k R^3}{3} \cos \theta \hat{r} + \frac{4\pi k R^3}{3} \sin \theta \hat{\theta} \]

\[ E = \frac{4\pi k R^3}{3} \frac{1}{r^2} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] \]

dipole field
Physical example with $\sigma(\theta) = k \cos \theta$

Two spheres of radii $R$, with equal but opposite uniform charge densities $\rho$ and $-\rho$, displaced by small distance $d \ll R$

Surface charge $\sigma$ builds up due to displacement. This is a uniformly "polarized" sphere.

Surface charge $\sigma': \sigma(\theta) = \rho Sr = \rho d \cos \theta$

\[ \sigma(\theta) = \rho d \cos \theta \]

Total dipole moment is $(\rho d)^2 \frac{4\pi}{3} R^3$

Polarization = \frac{\text{dipole moment}}{\text{volume}} = \rho d$

$\hat{E}$ field inside a uniformly polarized sphere is constant.

\[ \hat{E} = -\rho d \frac{4\pi}{3} \]
Grounded conducting sphere in uniform electric field $\mathbf{E} = E_0 \hat{z}$

as $r \to \infty$ far from sphere, $\mathbf{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$

boundary conditions

\[
\begin{align*}
\phi(r, \theta) &= 0 \\
\phi(r \to \infty, \theta) &= -E_0 r \cos \theta
\end{align*}
\]

Solution outside sphere has the form

\[\phi(r, \theta) = \sum_{\ell=0}^{\infty} \left[ A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos \theta) \]

From boundary condition as $r \to \infty$, we have

\[A_\ell = 0 \quad \text{all } \ell \neq 1\]

\[A_1 = -E_0 \quad \text{since } P_1(\cos \theta) = \cos \theta\]

\[\phi(r, \theta) = -E_0 r \cos \theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta)\]

From $\phi(r, \theta) = 0$ we have

\[0 = -E_0 R \cos \theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos \theta)\]

\[\Rightarrow B_\ell = 0 \quad \text{all } \ell \neq 1\]

\[B_1 = \frac{E_0 R}{R^2} \Rightarrow B_1 = +E_0 R^3\]
\[ \phi(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos \theta \]

1st term is just potential \(-E_0 r \cos \theta\) of the uniform applied electric field.

2nd term is potential due to the induced surface charge on the surface — it is a dipole field.

Induced charge density is

\[ 4\pi \sigma(\theta) = \left. \frac{\partial \Phi}{\partial r} \right|_{r=R} = E_0 \left( 1 + 2R^3 r^{-3} \right) \cos \theta = 3E_0 \cos \theta \]

\[ \sigma(\theta) = \frac{3}{4\pi} E_0 \cos \theta \] like uniformly polarized sphere \( k = \frac{3E_0}{4\pi} \)

From (2) we know that the field inside the sphere due to this \( \sigma \) is just

\[ -\frac{1}{2} \pi \hat{r} = -\frac{1}{2} \pi \frac{3E_0}{4\pi} \hat{r} = -E_0 \hat{r} \] This is just what is required so that the total field inside the conducting sphere vanishes.

Can check that outside the sphere, \( \vec{E} = -\hat{r} \Phi \) is normal to surface of sphere at \( r = R \).
Behavior of fields near a cylindrical hole at sharp tip

We now want to solve the $\nabla^2 \phi = 0$ with separation of variables, but now $\phi$ is restricted to range $0 \leq \theta \leq \beta$.

We still have azimuthal symmetry, but now, since we do not need solution to $\phi$ be finite for all $0 \leq \theta \leq \pi$, but only $\theta \in (0, \beta)$, we have more solutions to the $\Theta$ equation, i.e., $l$ does not have to be integer. It still needs $l > 0$ to be finite at $\theta = 0$.

See Jackson sec. 3.4 for details.
Multiple Expansions

Region with \( p \neq 0 \)

We want to find the potential \( \phi \) for an arbitrary localized distribution of charge \( \rho \), at distances far away \( r \gg R \).

\[
\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}
\]

**General Coulomb formula:**

We want an expansion of \( \frac{1}{|\vec{r} - \vec{r}'|} \) in powers of \( \frac{r'}{r} \) for \( r \gg r' \).

View this as the potential at \( \vec{r} \) due to a unit point charge located at position \( \vec{r}' \).

We take \( \vec{r}' \) on the \( \hat{z} \) axis.

\[
\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r}
\]

The problem has azimuthal symmetry \( \phi \) depends only on \( r \) and \( \theta \), so we can express it as an expansion in Legendre polynomials.

For \( r \gg r' \),

\[
\phi(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B_{2\ell}}{r^{\ell+1}} P_{2\ell}(\cos \theta)
\]

all \( A_\ell = 0 \)

as \( \ell \rightarrow \infty \)

\[
= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_{2\ell}}{r^{\ell}} P_{2\ell}(\cos \theta)
\]

as \( r \rightarrow \infty \)
We know \( \phi (r, \theta = 0) = \frac{1}{r - r'} \) (for \( r > r' \))

\[ \phi (r, 0) = \frac{1}{r} \sum_{\ell = 0}^{\infty} \frac{B_{\ell}}{r'^{\ell+1}} P_{\ell} (1) \]

\[ = \frac{1}{r} \sum_{\ell = 0}^{\infty} \frac{B_{\ell}}{r'^{\ell+1}} \] as \( P_{\ell} (1) = 1 \)

\[ = \frac{1}{r} \left( \frac{1}{1 - r/r'} \right) \approx \text{exact result from Coulomb} \]

Now Taylor expansion \( \frac{1}{1 - \epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots \)

\[ \Rightarrow \frac{1}{r} \sum_{\ell = 0}^{\infty} \frac{B_{\ell}}{r'^{\ell+1}} = \frac{1}{r} \left( 1 + \frac{r'}{r} + \left( \frac{r'}{r} \right)^2 + \left( \frac{r'}{r} \right)^3 + \cdots \right) \]

\[ \Rightarrow B_{\ell} = \left( \frac{r'}{r} \right)^{\ell} \] is solution

So for \( r > r' \)

\[ \frac{1}{|r - r'|} = \frac{1}{r} \sum_{\ell = 0}^{\infty} \left( \frac{r'}{r} \right)^{\ell} P_{\ell} (\cos \theta) \]

So for the charge distribution \( \rho \),

\[ \phi (r) = \int d^3 r' \frac{\rho (r')}{|r - r'|} = \int d^3 r' \frac{\rho (r')}{r} \sum_{\ell = 0}^{\infty} \left( \frac{r'}{r} \right)^{\ell} P_{\ell} (\cos \theta) \]

\[ = \sum_{\ell = 0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3 r' \rho (r') \left( \frac{r'}{r} \right)^{\ell} P_{\ell} (\cos \theta) \]

Where \( \theta \) is the angle between the fixed observation point \( r \) and the integration variable \( r' \).
This is the multipole expansion, which expresses the potential far from a localized source as a point series in \( r / r_0 \). It is exact provided one adds all the infinite \( l \) terms. In practice, one generally approximates by summing only up to some finite \( l \).

**Note:** in doing the integrals

\[ \int d^3 \vec{r} \int \frac{1}{(r_1^0)^l} \vec{r} \cdot \vec{e}_l \left( \cos \theta \right) \]

\( \theta \) is defined as the angle of \( \vec{r} \) with respect to observation point \( \vec{r}_0 \). We therefore in principle have to repeat the integration every time we change \( \vec{r}_0 \).

We will find a way around this by

(i) just looking explicitly at the few lowest order terms

(ii) a general method involving spherical harmonics \( Y_{lm} (\theta, \phi) \)