Suppose $\sigma(\theta) = k\cos\theta$ what is ϕ?

Note $\sigma(\theta) = k P_1(\cos\theta)$

hence only $A_1 \neq 0$ by orthogonality of $P_\ell(\cos\theta)$

$$A_1 = \frac{4\pi k}{2} \int_0^{\pi} d\theta \sin\theta \, P_1(\cos\theta) P_1(\cos\theta)$$

$$= \frac{4\pi k}{2}\left(\frac{2}{2+1}\right) = \frac{4}{3}\pi k$$

$$\Rightarrow \phi(r,\theta) = \begin{cases} \frac{4\pi}{3} k \, r\cos\theta & r < R \\ \frac{4\pi}{3} k \frac{R^3}{r^2} \cos\theta & r > R \end{cases}$$

we will see that potential outside the sphere is that of an ideal dipole with dipole moment

$$p = \frac{4}{3}\pi R^3 k$$

Inside the sphere, the potential $\phi = \frac{4\pi}{3} k z$ where $z = r\cos\theta$. The electric field inside the sphere is therefore the constant

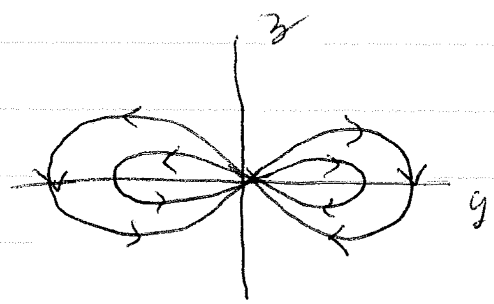
$$\vec{E} = -\vec{\nabla}\phi = -\frac{4}{3}\pi k \hat{z}$$

outside the sphere the field is

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

$$= \frac{8\pi}{3}k\frac{R^3}{r^3}\cos\theta\,\hat{r} + \frac{4\pi}{3}k\frac{R^3}{r^3}\sin\theta\,\hat{\theta}$$

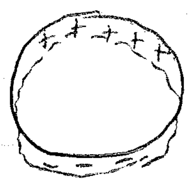
$$\vec{E} = \frac{4\pi}{3}R^3 k\frac{1}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$



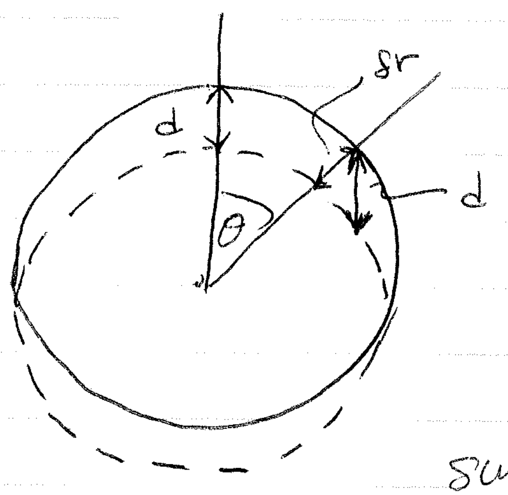
dipole field

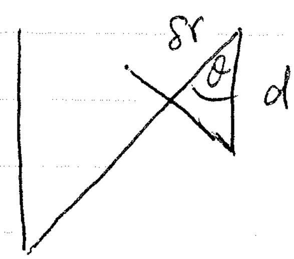
Physical exaple with $\sigma(\theta) = k\cos\theta$

Two spheres of radii R, with equal but opposite uniform charge densities ρ and $-\rho$, displaced by small distance $d \ll R$



surface charge σ builds up due to displacement
This is a uniformly "polarized" sphere





$d\cos\theta = \delta r$

surface charge is $\sigma(\theta) = \rho\,\delta r$
$= \rho d \cos\theta$

$$\boxed{\sigma(\theta) = \rho d \cos\theta}$$

total dipole moment is $(\rho d)\frac{4}{3}\pi R^3$

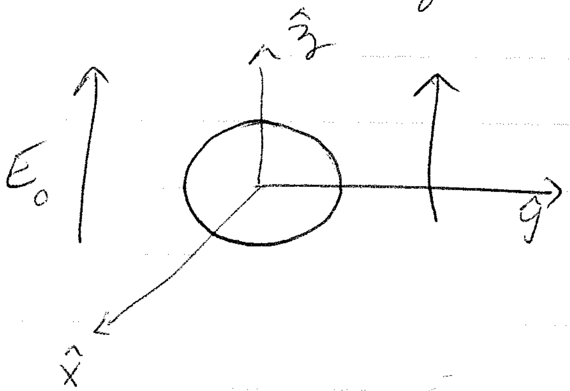
polarization $= \frac{\text{dipole moment}}{\text{volume}} = \rho d$

$\vec{E}$ field inside a uniformly polarized sphere is constant. $\vec{E} = -\rho d \frac{4\pi}{3}$

Grounded

(3) Conducting sphere in uniform electric field $\vec{E} = E_0 \hat{z}$

as $r \to \infty$ far from sphere, $\vec{E} = E_0 \hat{z} \Rightarrow \phi = -E_0 z$
$= -E_0 r \cos\theta$



boundary conditions

$$\begin{cases} \phi(R,\theta) = 0 \\ \phi(r\to\infty,\theta) = -E_0 r\cos\theta \end{cases}$$

solution outside sphere has the form

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta)$$

From boundary condition as $r \to \infty$ we have

$A_\ell = 0$ all $\ell \neq 1$

$A_1 = -E_0$ since $P_1(\cos\theta) = \cos\theta$

$$\phi(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

From $\phi(R,\theta) = 0$ we have

$$0 = -E_0 R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow B_\ell = 0 \text{ all } \ell \neq 1$$

$$\frac{B_1}{R^2} = E_0 R \Rightarrow B_1 = +E_0 R^3$$

So $$\boxed{\phi(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta}$$

1^{st} term is just potential $-E_0 r\cos\theta$ of the uniform applied electric field.

2^{nd} term is potential due to the induced surface charge on the surface - it is a dipole field

Induced charge density is

$$4\pi\sigma(\theta) = -\left.\frac{\partial\phi}{\partial r}\right|_{r=R} = E_0\left(1 + \frac{2R^3}{R^3}\right)\cos\theta$$

$$= 3E_0\cos\theta$$

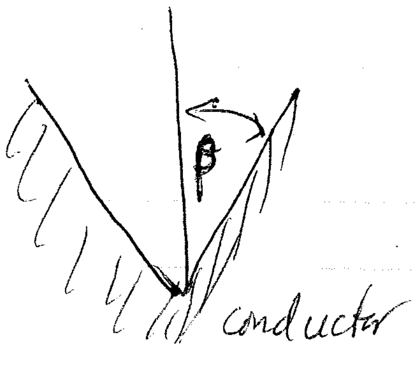
$$\sigma(\theta) = \frac{3}{4\pi}E_0\cos\theta$$ like uniformly polarized sphere $k = \frac{3E_0}{4\pi}$

from ② we know that the field inside the sphere due to this σ is just $-\frac{4}{3}\pi k\hat{z} = -\frac{4}{3}\pi\frac{3E_0}{4\pi}\hat{z}$

$= -E_0\hat{z}$. This is just what is required so that the total field in the conducting sphere vanishes.

Can check that outside the sphere, $\vec{E} = -\vec{\nabla}\phi$ is normal to surface of sphere at $r = R$.

Behavior of fields near conical hole or sharp tip



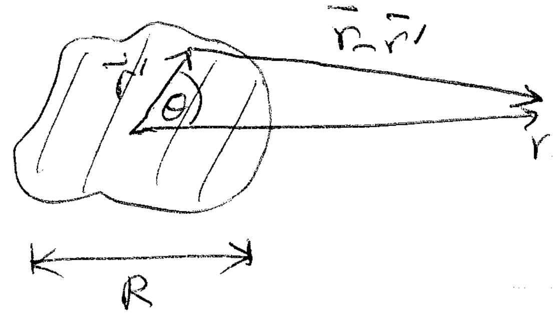
We now want to solve the $\nabla^2\phi = 0$ with separation of variables, but now θ is restricted to range $0 \le \theta \le \beta$.

We still have azimuthal symmetry, but now, since we do not need solution to be finite for all $\theta \in [0, \pi]$, but only $\theta \in (0, \beta)$, we have more solutions to the Θ equation, i.e. l does not have to be integer. - still need $l \geq 0$ to be finite at $\theta = 0$.

see Jackson sec. 3.4 for details.

Multipole Expansion

region with $\rho \neq 0$

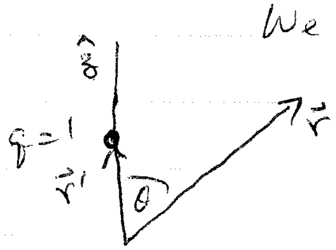


We want to find the potential ϕ for an arbitrary localized distribution of charge ρ, at distances far away $r >> R$.

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad \text{General Coulomb formula}$$

we want an expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r >> r'$

$\frac{1}{|\vec{r}-\vec{r}'|}$ view this as the potential at $\vec{r}$ due to a unit point charge located at position $\vec{r}'$. We take $\vec{r}'$ on the $\hat{z}$ axis.



The problem has azimuthal symmetry $\Rightarrow \phi$ depends only on r and θ, so we can express it as an expansion in Legendre polynomials.

For $r > r'$,

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

all $A_\ell = 0$ as need $\phi \to 0$ as $r \to \infty$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} P_\ell(\cos\theta)$$

We know $\phi(r, \theta=0) = \dfrac{1}{r-r'}$ (for $r > r'$)

← scalars here since when $\theta = 0$, $\vec{r}$ and $\vec{r}'$ are both on $\hat{z}$ axis

$$\Rightarrow \phi(r,0) = \frac{1}{r}\sum_{\ell} \frac{B_\ell}{r^\ell} P_\ell(1)$$

$$= \frac{1}{r}\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} \quad \text{as } P_\ell(1) = 1$$

$$= \frac{1}{r}\frac{1}{(1 - r'/r)} \quad \leftarrow \text{exact result from Coulomb}$$

Now Taylor expansion $\dfrac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 + \cdots$

$$\Rightarrow \frac{1}{r}\sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} = \frac{1}{r}\left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \left(\frac{r'}{r}\right)^3 + \cdots\right)$$

$\Rightarrow$ $B_\ell = (r')^\ell$ is solution

So for $r > r'$

$$\boxed{\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)}$$

So for the charge distribution ρ,

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \int d^3r' \frac{\rho(\vec{r}')}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \rho(\vec{r}') (r')^\ell P_\ell(\cos\theta)$$

where θ is the angle between the fixed observation point $\vec{r}$ and the integration variable $\vec{r}'$.

This is the multipole expansion, which expresses the potential far from a localized source as a power series in (r'/r). It is *exact* provided one adds all the infinite l terms. In practice, one generally approximates by summing only up to some finite l.

Note: in doing the integrals

$$\int d^3r' \, \rho(\vec{r}')(r')^l P_l(\cos\theta)$$

θ is defined as the angle of $\vec{r}'$ with respect to observation point $\vec{r}$. We therefore in principle have to repeat this integration every time we change $\vec{r}$.

We will find a way around this by
(i) first looking explicitly at the few lowest order terms
(ii) a general method involving spherical harmonics $Y_{lm}(\theta, \varphi)$