

## Magnetic Materials

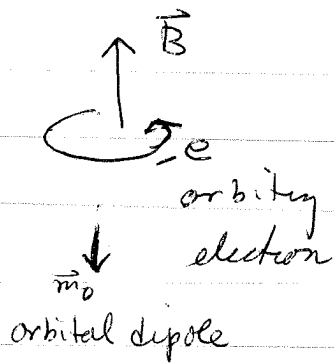
Circulating currents on atomic scale give rise to local magnetic dipole moments, which create local magnetic fields in the material.

Sources of circulating atomic currents:

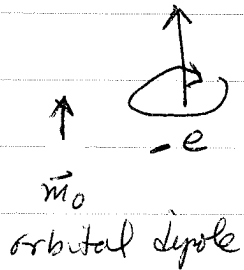
- 1) intrinsic angular momentum of electrons, i.e. "electron spin" - can add up and give a net angular momentum to atom
- 2) orbital angular momentum of electrons - can add up to give net angular momentum of atom.

(1) + (2)  $\Rightarrow$  atoms can have a net magnetic dipole moment. When  $\vec{B} = 0$ , these atomic moments are generally in random orientations <sup>and average to zero</sup> (exception is a ferromagnet where moments can align even if  $\vec{B} = 0$ )  
When apply  $\vec{B} \neq 0$ , the moments tend to align parallel to  $\vec{B}$  giving a net magnetization density  $\vec{M} \propto \vec{B}$ . This is a paramagnetic effect.

But there is also a diamagnetic effect from orbital angular momentum (exists even if total angular momentum of electrons is zero, i.e. exists for atoms with zero net dipole moment)



← applying  $\vec{B}$  to orbiting electron speeds up its orbital velocity. Increased angular momentum of negatively charged electron gives change in dipole moment  $\Delta \vec{m} \propto -\vec{B}$



← applying  $\vec{B}$  to orbiting electron slows down its orbital velocity. Net result is again that  $\Delta \vec{m} \propto -\vec{B}$

see Griffiths  
chpt 6 + prob  
7-17 2nd ed  
for details

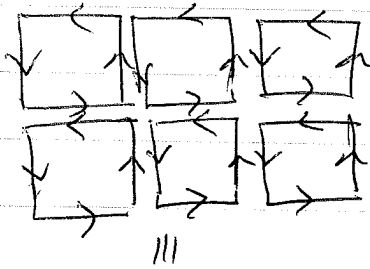
No matter which way electron orbits with respect to  $\vec{B}$ , result is a decrease in magnetic moment, so  $\Delta \vec{m} \propto -\vec{B}$ . That  $\Delta \vec{m}$  is opposite to  $\vec{B}$  is called diamagnetism.

Model atomic magnetic moments as small current loops. When loops get oriented, there is non zero average magnetization density

$$\vec{M}(\vec{r}) = \sum_i \vec{m}_i \delta(\vec{r} - \vec{r}_i)$$

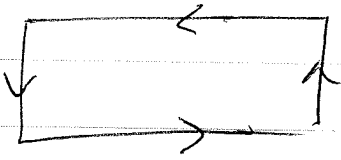
then net effect is to have a current flowing around the system. This current gives rise to magnetic fields

aligned atomic moments in a uniform applied  $\vec{B}$



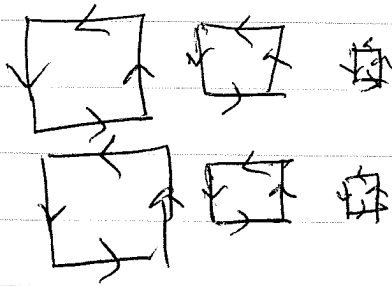
in interior, currents in opposite directions cancel also  $\vec{j} = 0$  inside

⊙  $\vec{B}$  out of page



but is net circulation of current around boundary of material  
 $\Rightarrow$  surface current  $\vec{K}_{\text{bound}}$

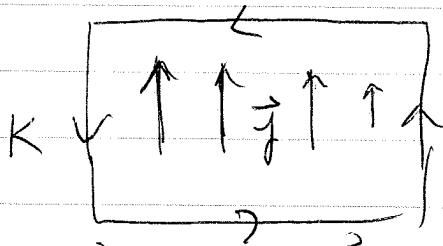
If  $\vec{B}$  is not uniform, then  $\vec{M}$  is not uniform  
 Can create finite current density  $\vec{j}$  in interior, as well as surface currents



Now currents in interior do not cancel. Net current  $\vec{j}_{\text{bound}}$  in interior

$\vec{B}$  strong

$\vec{B}$  weak



$\vec{M}_{\text{strong}}$

$\vec{M}_{\text{weak}}$

$\vec{B}$  out of page  $\Rightarrow \vec{M}$  out of page  
 ~~$\vec{M}$  varies along page~~  
 $\vec{M}$  varies in direction  $\perp$  direction of  $\vec{M}$   
 $\Rightarrow \nabla \times \vec{M} \neq 0$  gives  $\vec{j}_{\text{bound}}$

## Average current

$$\langle \vec{j}_0 \rangle = \left\langle \sum_{i \in \text{free}} q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i) \right\rangle + \sum_n \langle \vec{j}_n \rangle$$

↑
↑  
 current from free charges      current from molecule n of the dielectric

$$\begin{aligned} \langle \vec{j}_n(\vec{r}, t) \rangle &= \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \langle \delta(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t)) \rangle \\ &= \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) f(\vec{r} - \vec{r}_n(t) - \vec{r}_{ni}(t)) \end{aligned}$$

↑
↑
↑
↑  
 $\vec{v}_n = \frac{d\vec{r}_n}{dt}$      $\vec{v}_{ni} = \frac{d\vec{r}_{ni}}{dt}$     position of CM of molec n    position of charge i wrt CM

as with  $\langle j_0 \rangle$ , we can expand in  $\vec{r}_{ni}$

$$\begin{aligned} \langle \vec{j}_n \rangle &= \sum_{i \in n} q_i (\vec{v}_n + \vec{v}_{ni}) \left\{ f(\vec{r} - \vec{r}_n) - \vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n) \right. \\ &\quad \left. + \frac{1}{2} \sum_{\alpha\beta} (r_{ni})_\alpha (r_{ni})_\beta \frac{\partial^2 f(\vec{r} - \vec{r}_n)}{\partial r_\alpha \partial r_\beta} \right. \\ &\quad \left. + \dots \right\} \end{aligned}$$

we will keep only the first two terms in the expansion

The various terms we have to consider are

$$\textcircled{1} \quad \sum_{i \in n} q_i \vec{v}_n f(\vec{r} - \vec{r}_n)$$

$$\textcircled{2} \quad \sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r} - \vec{r}_n)$$

$$\textcircled{3} \quad - \sum_{i \in n} q_i \vec{v}_n \left[ \vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n) \right]$$

$$\textcircled{4} \quad - \sum_{i \in n} q_i \vec{v}_{ni} \left[ \vec{r}_{ni} \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n) \right]$$

$$\textcircled{1} = \vec{v}_n f(\vec{r} - \vec{r}_n) \sum_{i \in n} q_i = g_n \vec{v}_n f(\vec{r} - \vec{r}_n) \\ = \langle g_n \vec{v}_n \delta(\vec{r} - \vec{r}_n) \rangle$$

this is just current of molecule as if it were a point charge  $g_n$ . For a neutral molecule  $g_n = 0$  and this term vanishes.

$$\textcircled{2} \quad \text{Note: } \frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r} - \vec{r}_n) \rangle = \frac{\partial}{\partial t} \left( \sum_{i \in n} q_i \vec{r}_{ni} f(\vec{r} - \vec{r}_n) \right) \\ = \sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r} - \vec{r}_n) \\ + \sum_{i \in n} q_i \vec{r}_{ni} \left[ -\vec{\nabla} f(\vec{r} - \vec{r}_n) \cdot \vec{v}_n \right]$$

$$\text{So for } \textcircled{2}, \quad \sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r} - \vec{r}_n)$$

$$= \frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r} - \vec{r}_n) \rangle$$

$$+ \left[ \vec{v}_n \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n) \right] \vec{p}_n$$

$S_0$

$$\textcircled{2} = \sum_{i \in n} q_i \vec{v}_{ni} f(\vec{r} - \vec{r}_n) = \frac{\partial}{\partial t} \langle \vec{p}_n \delta(\vec{r} - \vec{r}_n) \rangle + (\vec{v}_n \cdot \vec{\nabla}) \langle \vec{p}_n \delta(\vec{r} - \vec{r}_n) \rangle$$

2nd term is  $\sum_{\alpha} v_{n\alpha} \frac{\partial}{\partial r_{\alpha}} \langle \vec{p}_n \delta(\vec{r} - \vec{r}_n) \rangle$

$$\textcircled{3} = -\vec{v}_n \left( \sum_{i \in n} q_i \vec{r}_{ni} \right) \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n) = -\vec{v}_n \cdot (\vec{p}_n \cdot \vec{\nabla} f(\vec{r} - \vec{r}_n))$$

$$= -\vec{v}_n \cdot \vec{\nabla} \cdot \langle \vec{p}_n \delta(\vec{r} - \vec{r}_n) \rangle = \sum_{\alpha} \vec{v}_n \frac{\partial}{\partial r_{\alpha}} \langle p_{n\alpha} \delta(\vec{r} - \vec{r}_n) \rangle$$

$$\textcircled{4} = -\vec{\nabla} f(\vec{r} - \vec{r}_n) \cdot \sum_{i \in n} q_i \vec{r}_{ni} \vec{v}_{ni}$$

We have seen the tensor  $\sum_{i \in n} q_i \vec{r}_{ni} \vec{v}_{ni}$  before when we considered the magnetic dipole moment

$$\sum_{i \in n} q_i \vec{r}_{ni} \vec{v}_{ni} = \int d^3r \vec{r} \vec{j} \quad \text{where } \vec{j}(\vec{r}) \equiv \sum_{i \in n} q_i \vec{v}_{ni} \delta(\vec{r} - \vec{r}_{ni})$$

is current density with respect to center of mass of molecule

We had  $\int d^3r \vec{r} \vec{j} = -\int d^3r \vec{j} \vec{r} - \int d^3r (\vec{\nabla} \cdot \vec{j}) \vec{r} \vec{r}$

↑  
in statics,  $\vec{\nabla} \cdot \vec{j} = 0$

in general  $\vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$

$$\int d^3r \vec{r} \vec{j} = -\int d^3r \vec{j} \vec{r} + \int d^3r \frac{\partial \rho}{\partial t} \vec{r} \vec{r}$$

$$= -\int d^3r \vec{j} \vec{r} + \frac{\partial}{\partial t} \left[ \int d^3r \rho \vec{r} \vec{r} \right]$$

↑

although this is not zero, it is a quadrupole term of the same order as the terms we dropped when we truncated

$$\sim O\left(\frac{a_0}{L}\right)^2$$

$$\text{So } \int d^3r \vec{r} \vec{j} \approx - \int d^3r \vec{j} \vec{r} \quad \text{ignoring the quadrupole term}$$

$$= \frac{1}{2} \int d^3r [ \vec{r} \vec{j} - \vec{j} \vec{r} ]$$

$$\sum_{i \in n} q_i \vec{r}_{ni} \vec{v}_{ni} = \frac{1}{2} \sum_{i \in n} q_i [ \vec{r}_{ni} \vec{v}_{ni} - \vec{v}_{ni} \vec{r}_{ni} ]$$

$$- \vec{\nabla} f(\vec{r} - \vec{r}_n) \cdot \sum_{i \in n} q_i \vec{r}_{ni} \vec{v}_{ni} = - \vec{\nabla} f(\vec{r} - \vec{r}_n) \cdot \frac{1}{2} \sum_{i \in n} q_i [ \vec{r}_{ni} \vec{v}_{ni} - \vec{v}_{ni} \vec{r}_{ni} ]$$

$$= \frac{-1}{2} \sum_{i \in n} q_i [ (\vec{\nabla} f \cdot \vec{r}_{ni}) \vec{v}_{ni} - (\vec{\nabla} f \cdot \vec{v}_{ni}) \vec{r}_{ni} ]$$

$$= -\frac{1}{2} \sum_{i \in n} q_i \vec{\nabla} f \times (\vec{v}_{ni} \times \vec{r}_{ni}) \quad \text{triple product rule}$$

$$= \vec{\nabla} f(\vec{r} - \vec{r}_n) \times \frac{1}{2} \sum_{i \in n} \vec{r}_{ni} \times \vec{v}_{ni} q_i$$

$$= \vec{\nabla} f(\vec{r} - \vec{r}_n) \times \frac{1}{2} \int d^3r \vec{r} \times \vec{j}$$

$$= \vec{\nabla} f(\vec{r} - \vec{r}_n) \times c \vec{m}_n \quad \text{where } \vec{m}_n = \frac{1}{2c} \sum_{i \in n} \vec{r}_{ni} \times \vec{v}_{ni} q_i$$

↳ magnetic dipole moment of molecule n

$$= \vec{\nabla} \times f(\vec{r} - \vec{r}_n) c \vec{m}_n$$

$$= \vec{\nabla} \times \langle c \vec{m}_n \delta(\vec{r} - \vec{r}_n) \rangle$$

Adding all the pieces

$$\begin{aligned}
 \langle \vec{j}_n \rangle = & \underbrace{\langle g_n \vec{v}_n \delta(\vec{r}-\vec{r}_n) \rangle}_{(1)} + c \vec{\nabla} \times \underbrace{\langle \vec{m}_n \delta(\vec{r}-\vec{r}_n) \rangle}_{(4)} \\
 & + \frac{\partial}{\partial t} \underbrace{\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle}_{(2)} + (\vec{v}_n \cdot \vec{\nabla}) \underbrace{\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle}_{(2)} \\
 & - \vec{v}_n \cdot \vec{\nabla} \underbrace{\langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle}_{(3)}
 \end{aligned}$$

Define  $\vec{M}(\vec{r}) \equiv \sum_n \langle \vec{m}_n \delta(\vec{r}-\vec{r}_n) \rangle$  average magnetization density

$\vec{P}(\vec{r}) \equiv \sum_n \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle$  polarization density, as before

$$\begin{aligned}
 \sum_n \langle \vec{j}_n \rangle = & \sum_n \langle g_n \vec{v}_n \delta(\vec{r}-\vec{r}_n) \rangle + c \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \\
 & + \sum_n \left[ (\vec{v}_n \cdot \vec{\nabla}) \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle - \vec{v}_n \cdot \vec{\nabla} \langle \vec{p}_n \delta(\vec{r}-\vec{r}_n) \rangle \right]
 \end{aligned}$$

see Jackson (6.96) for additional electric quadrupole terms

The last term on the right hand side is usually small and ignored. This is because the molecular velocities  $\vec{v}_n$  are usually small, and randomly oriented, so that they average to zero. (see Jackson (6.100) for case of net translation of dielectric,  $\vec{v}_n = \text{const}$  all  $n$ )



Define macroscopic current density

$$\vec{j}(\vec{r}, t) = \left\langle \sum_{i \in \text{free}} q_i \vec{v}_i \delta(\vec{r} - \vec{r}_i) \right\rangle + \left\langle \sum_n q_n \vec{v}_n \delta(\vec{r} - \vec{r}_n) \right\rangle$$

↑
↑  
 current of free charges      current of molecular drifting  
 if molecules are charged

Then  $\langle \vec{j}_0 \rangle = \vec{j} + c \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$

Ampere's law becomes upon averaging

$$\begin{aligned} \vec{\nabla} \times \vec{B} &= 4\pi \langle \vec{j}_0 \rangle + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \\ &= \frac{4\pi}{c} \vec{j} + 4\pi \vec{\nabla} \times \vec{M} + \frac{4\pi}{c} \frac{\partial \vec{P}}{\partial t} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

$$\vec{\nabla} \times (\vec{B} - 4\pi \vec{M}) = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} (\vec{E} + 4\pi \vec{P})$$

define  $\vec{H} \equiv \vec{B} - 4\pi \vec{M}$  to get

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t}$$

$$\vec{D} = \vec{E} + 4\pi \vec{P} \text{ as before}$$

official nomenclature:  $\vec{B}$  is the magnetic induction

$\vec{H}$  is the magnetic field

common usage: both ~~H~~ and  $\vec{B}$  are called magnetic field