Linear Materials

Macroscopic Maxwell Equations

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \qquad \vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

where ρ and $\vec{j}$ are macroscopic charge + current densities and

$$\vec{D} = \vec{E} + 4\pi\vec{P} \qquad \vec{P} \text{ is polarization density}$$
$$\vec{H} = \vec{B} - 4\pi\vec{M} \qquad \vec{M} \text{ is magnetization density}$$

To close these equations, we will in general need to know how $\vec{P}$ and $\vec{M}$ are related to the $\vec{E}$ and $\vec{B}$ in the material.

In some materials, there can be a finite $\vec{P}$ or $\vec{M}$ even if $\vec{E}$ and $\vec{B}$ are zero:

<u>Ferromagnet</u>: $\vec{M}$ can be non zero even if $\vec{B}=0$

<u>Ferroelectric</u>: $\vec{P}$ can be non zero even if $\vec{E}=0$

But more common are <u>linear</u> materials in which, for <u>small</u> $\vec{E}$ and $\vec{B}$, one has $\vec{P}\propto\vec{E}$ and $\vec{M}\propto\vec{B}$.

linear dielectric

$$\vec{P} = \chi_e \vec{E}$$ χ_e is "electric susceptibility"

$\chi_e > 0$ for statics

$$\vec{D} = \vec{E} + 4\pi\vec{P} = (1 + 4\pi\chi_e)\vec{E}$$

$$\vec{D} = \epsilon\vec{E} \quad \text{with} \quad \epsilon = 1 + 4\pi\chi_e$$

ϵ is the dielectric constant

linear magnetic materials

$$\vec{M} = \chi_m \vec{H}$$ χ_m is "magnetic susceptibility"

$\chi_m > 0 \Rightarrow$ paramagnetic

$\chi_m < 0 \Rightarrow$ diamagnetic

$$\vec{H} = \vec{B} - 4\pi\vec{M} = \vec{B} - 4\pi\chi_m\vec{H}$$

$$\vec{B} = (1 + 4\pi\chi_m)\vec{H}$$

$$\vec{B} = \mu\vec{H} \quad \text{with} \quad \mu = 1 + 4\pi\chi_m$$

μ is magnetic permeability

For statics, $\chi_e > 0$ and χ_m (or alternatively ϵ and μ) are constants depending on the material.

When we consider dynamics we will see that ϵ becomes a function of frequency.

Clausius - Mossotti equation

Electric susceptibility & atomic polarizability

If a field $\vec{E}_{loc}$ is applied to an atom, it gets polarized

$$\vec{p} = \alpha \vec{E}_{loc}$$

atomic dipole moment   atomic polarizability   "local field" - field the atom sees

α is what one calculates from a microscopic theory

If $\vec{E}_{loc} = \vec{E}$ the average field in the material then electric susceptibility given by

$$\vec{P} = n\vec{p} = n\alpha \vec{E}_{loc} = n\alpha \vec{E} = \chi_e \vec{E}$$

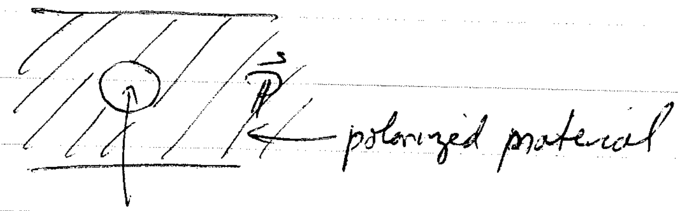
$\Rightarrow \chi_e = n\alpha$ where n = density of atoms

But a more careful consideration shows $\vec{E}_{loc} \neq \vec{E}$

The average field $\vec{E}$ includes the electric field created by the polarized atom itself. $\vec{E}_{loc}$, the local field the atom sees, should exclude its own self field.

$$\vec{E} = \vec{E}_{loc} + \vec{E}_{atom}$$

average field   average field excluding atom   average field of the atom



cut out sphere whose volume is $1/n$ the volume per atom

$\vec{E}_{loc}$ is field <u>excluding</u> the field of the polarized sphere of volume $1/n$.

$\vec{E}_{atom}$ is field of the polarized sphere

$$E_{atom} = -\frac{4\pi\vec{\mathbb{P}}}{3} = -\frac{4\pi}{3} n\vec{p}$$

$$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi}{3}\vec{\mathbb{P}} = \vec{E} + \frac{4\pi}{3} n\vec{p}$$

$$\vec{p} = \alpha\vec{E}_{loc} = \alpha\left(\vec{E} + \frac{4\pi}{3} n\vec{p}\right) = \alpha\vec{E} + \frac{4\pi}{3} n\alpha\,\vec{p}$$

$$\vec{p} = \frac{\alpha}{1 - \frac{4\pi}{3} n\alpha}\vec{E}$$

$$\vec{\mathbb{P}} = n\vec{p} = \frac{\alpha n}{1 - \frac{4\pi}{3} n\alpha}\vec{E} = \chi_e\vec{E}$$

$$\boxed{\chi_e = \frac{n\alpha}{1 - \frac{4\pi}{3} n\alpha}}$$

or solve for α in terms of ϵ

$$\chi_e = \frac{n\alpha}{1-\frac{4\pi}{3}n\alpha} \Rightarrow \chi_e - \frac{4\pi}{3}n\alpha\chi_e = \alpha n$$

$$\Rightarrow \alpha = \frac{\chi_e}{n(1+\frac{4\pi}{3}\chi_e)}$$

$$\epsilon = 1+4\pi\chi_e \Rightarrow \alpha = \frac{\epsilon-1}{4\pi n}\frac{1}{\left(1+\frac{\epsilon-1}{3}\right)}$$

relates atomic polarizability to measured dielectric constant

$$\boxed{\alpha = \frac{3}{4\pi n}\left(\frac{\epsilon-1}{\epsilon+2}\right)}$$

Clausius Mossotti or Lorentz-Lorenz equation

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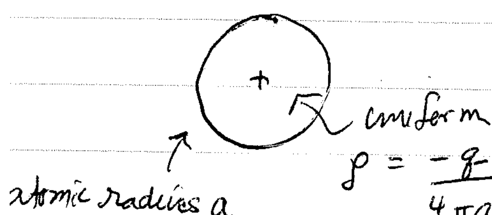
simple model for α



$\rho = \frac{-q}{\frac{4}{3}\pi a^3}$

field inside is $E(r) = \frac{4\pi\rho}{3} r\,\hat{r}$

~~polarization~~

In external field E_0, net forces balance $\Rightarrow qE_0 = q\frac{4\pi\rho}{3}d$

$$p = qd = \frac{3}{4\pi\rho}qE_0 = \frac{3\left(\frac{4}{3}\pi a^3\right)}{4\pi q}qE_0$$

$$= a^3 E_0 \Rightarrow \boxed{\alpha = a^3}$$

$$\chi_e = \frac{na^3}{1-\frac{4\pi}{3}na^3}$$

if $f = n\frac{4}{3}\pi a^3$ fraction of vol that is occupied by atoms

$$\boxed{\chi_e = \frac{1}{4\pi}\frac{3f}{1-f}}$$

Linear dielectrics

bound charge is proportional to free charge

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(\chi_e\vec{E}) = -\vec{\nabla}\cdot\left(\frac{\chi_e}{\epsilon}\vec{D}\right)$$

if χ_e (and hence ϵ) is spatially constant, then

$$\rho_b = -\frac{\chi_e}{\epsilon}\vec{\nabla}\cdot\vec{D} = -\frac{\chi_e}{\epsilon}4\pi\rho$$

$$\boxed{\rho_b = -\frac{4\pi\chi_e}{1+4\pi\chi_e}\rho}$$

when free charge $\rho = 0$, then $\rho_b = 0$

$$\rho_{total} = \rho + \rho_b = \rho\left[1 - \frac{4\pi\chi_e}{1+4\pi\chi_e}\right] = \frac{\rho}{1+4\pi\chi_e} = \boxed{\frac{\rho}{\epsilon} = \rho_{total}}$$

bound charge "screens" the free charge so the total charge is reduced compared to the free charge.

For linear dielectrics

Statics

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{E} = 0$$

$$\vec{D} = \epsilon\vec{E} \Rightarrow \vec{\nabla}\cdot(\epsilon\vec{E}) = 4\pi\rho$$

if ϵ is constant in space then $\epsilon\,\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{E} &= 4\pi\rho/\epsilon = 4\pi\rho_{tot}\\ \vec{\nabla}\times\vec{E} &= 0\end{aligned}\right\}$$ look just like ordinary electrostatics but with $\rho \to \rho/\epsilon$

Alternatively, could write $\vec{E} = \vec{D}/\epsilon$

$$\Rightarrow \vec{\nabla}\times(\vec{D}/\epsilon) = 0$$
$$\Rightarrow \vec{\nabla}\times\vec{D} = 0$$ when ϵ constant in space

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{D} &= 4\pi\rho\\ \vec{\nabla}\times\vec{D} &= 0\end{aligned}\right\}$$ looks just like ordinary electrostatics, but with $\vec{E}\to\vec{D}$

Complication arises at interface between dielectrics (or between dielectric and vacuum). At interface, ϵ is not constant $\Rightarrow \vec{\nabla}\times\vec{D} \neq 0$.

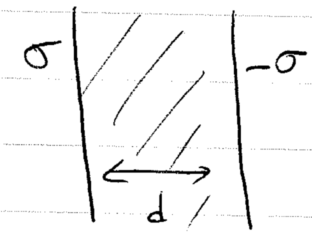
What we can do is to solve for $\vec{E}$ or $\vec{D}$ inside each dielectric separately, and then use the boundary conditions

$$\hat{n}\cdot(\vec{D}^{above} - \vec{D}^{below}) = 4\pi\sigma$$
$$\hat{t}\cdot(\vec{E}^{above} - \vec{E}^{below}) = 0$$

to match solutions across the interfaces.

A similar story holds for linear magnetic materials

Simple example: parallel plate capacitor filled with a dielectric



σ free charge

What is E between plates?

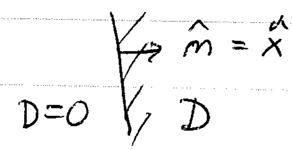
we know $\vec{E} = \vec{D} = 0$ outside plates

Between plates $\vec{\nabla} \cdot \vec{D} = 0$ as $\rho = 0$

$\vec{D} = D(x)\,\hat{x} \quad \Rightarrow \quad \dfrac{\partial D}{\partial x} = 0 \Rightarrow D$ is constant

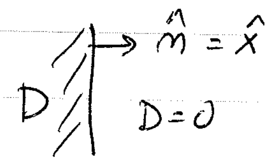
Boundary conditions:

left side plate



$$\hat{x} \cdot (\vec{D}^{\text{above}} - \vec{D}^{\text{below}}) = D = 4\pi\sigma$$

right side plate



$$\hat{x} \cdot (\vec{D}^{\text{above}} - \vec{D}^{\text{below}}) = -D = 4\pi(-\sigma)$$

$D = 4\pi\sigma$ as before

$$\Rightarrow \quad \vec{D} = 4\pi\sigma\,\hat{x}$$

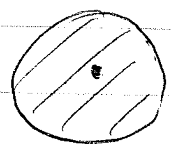
$$\boxed{\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{4\pi\sigma}{\epsilon}\,\hat{x}}$$

electric field reduced by factor $1/\epsilon$ as compared to capacitor with vacuum between plates

see Jackson section 4.4 for more interesting examples
– dielectric sphere in uniform applied $\vec{E}$

see Jackson section $\binom{5.11}{5.12}$ for an interesting magnetic b.c. problem

point charge within a dielectric sphere



pt charge q at center of dielectric sphere of radius R, dielectric const ϵ

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad = \quad \oint_S da\, \hat{n}\cdot\vec{D} = 4\pi Q_{encl}$$

From symmetry $\vec{D}(\vec{r}) = D(r)\,\hat{r}$

$$\oint_S da\, \hat{n}\cdot\vec{D} = 4\pi r^2 D(r) = 4\pi q$$

sphere of radius r $\nearrow$

$$\vec{D} = \frac{q}{r^2}\hat{r} \qquad \text{all } r$$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \dfrac{q}{\epsilon r^2}\hat{r} & r < R \\[2ex] \dfrac{q}{r^2}\hat{r} & r > R \end{cases}$$

can check that tangential component of $\vec{E}$ is continuous and normal component of $\vec{D}$ is continuous as there is no free σ at surface of dielectric.

normal component of $\vec{E}$ jumps by

$$\hat{n}\cdot\left(\vec{E}^{above} - \vec{E}^{below}\right) = \frac{q}{R^2} - \frac{q}{\epsilon R^2} = \frac{q}{R^2}\left(1 - \frac{1}{\epsilon}\right) = \frac{q}{R^2}\left(\frac{\epsilon - 1}{\epsilon}\right)$$

$$= \frac{q}{R^2}\left(\frac{4\pi\chi_e}{1+4\pi\chi_e}\right) = 4\pi\sigma_{total} = 4\pi\sigma_b$$

$$\Rightarrow \sigma_b = \frac{q}{4\pi R^2}\left(\frac{4\pi\chi_e}{1+4\pi\chi_e}\right) = \frac{q\,\chi_e}{R^2\epsilon}$$

We can check this directly

$$\vec{P} = \chi_e \vec{E} = \frac{\chi_e}{\epsilon} \frac{q}{r^2} \hat{r}$$

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{\chi_e}{\epsilon} q \; 4\pi \delta(\vec{r})$$

bound charge at origin $q_b = -\frac{\chi_e}{\epsilon} 4\pi q$

total charge at origin is $q + q_b = q\left(1 - \frac{4\pi \chi_e}{\epsilon}\right)$

$\epsilon = 1 + 4\pi \chi_e$ $\qquad = q\left(\frac{\epsilon - 4\pi\chi_e}{\epsilon}\right) = \frac{q}{\epsilon}$ screened charge

at surface,

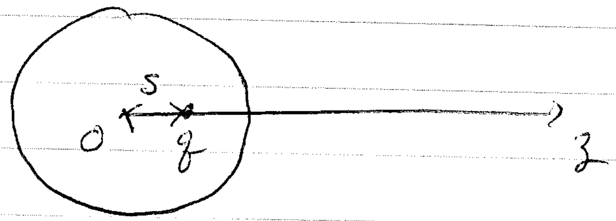
$$\sigma_b = \hat{n} \cdot \vec{P} = \frac{\chi_e}{\epsilon} \frac{q}{R^2}$$

agrees with what we get from jump in $\hat{n} \cdot \vec{E}$.

Note: inside the dielectric the $\vec{E}$ field is that of the screened point charge $\frac{q}{\epsilon}$

outside the dielectric $\vec{E}$ is just that of the free charge q - there is no evidence in $\vec{E}_{out}$ that the dielectric even exists!

Now consider same problem but q is off center



what is $\vec{E}$ inside + outside?

inside $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ where $\rho = q\,\delta(\vec{r}-s\hat{z})$

$$\vec{D} = \epsilon\vec{E} \Rightarrow \vec{\nabla}\cdot\vec{E} = 4\pi\rho/\epsilon$$
$$\vec{E} = -\vec{\nabla}\phi \Rightarrow \nabla^2\phi = -\frac{4\pi\rho}{\epsilon} = -\frac{4\pi q}{\epsilon}\,\delta(\vec{r}-s\hat{z})$$

solution for ϕ will be of the form

$$\phi(\vec{r}) = \frac{q}{\epsilon|\vec{r}-s\hat{z}|} + F(\vec{r})$$

where 1st term is due to the point charge q/ϵ and 2nd term satisfies $\nabla^2 F = 0$ and will be chosen to get the correct behavior at the boundary of the dielectric

Since there is azimuthal symmetry about $\hat{z}$ we can write

$$F(\vec{r}) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

there are no $\frac{b_l}{r^{l+1}}$ terms since F should not diverge at the origin

So inside, $r < R$

$$\phi^{in}(\vec{r}) = \frac{q}{\epsilon|\vec{r} - s\hat{z}|} + \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta)$$

From our discussion of electric multipole expansion, we know we can write for $r > s$,

$$\frac{1}{|\vec{r} - s\hat{z}|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{s}{r}\right)^\ell P_\ell(\cos\theta)$$

So for $\underline{\underline{r > s}}$ (not true for $r < s$!)

$$\phi^{in}(\vec{r}) = \sum_{\ell=0}^{\infty}\left(\frac{q}{\epsilon r}\left(\frac{s}{r}\right)^\ell + a_\ell r^\ell\right) P_\ell(\cos\theta)$$

Outside the sphere there is no charge, so $\vec{\nabla}\cdot\vec{E} = 0$ or $\nabla^2\phi = 0$

$$\Rightarrow \phi^{out}(\vec{r}) = \sum_{\ell=0}^{\infty}\frac{b_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

there are no $a_\ell r^\ell$ terms since $\phi^{out} \to 0$ as $r \to \infty$

To determine the unknown a_ℓ and b_ℓ we use the boundary conditions at surface of dielectric at $r = R$

① <u>tangential component</u> $\vec{E}$ is continuous

$$\vec{E} = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} = E_r\hat{r} + E_\theta\hat{\theta}$$

$\Rightarrow$ E_θ is continuous at $r = R$

condition that E_θ is continuous is the same condition that ϕ is continuous (check this out for yourself if you are not sure)

as $\vec{E}^{above} - \vec{E}^{below} = 4\pi\sigma_{tot}\hat{n}$

$$\Rightarrow \phi^{in}(R,\theta) = \phi^{out}(R,\theta)$$

$$\frac{q}{\epsilon R}\left(\frac{s}{R}\right)^l + a_l R^l = \frac{b_l}{R^{l+1}}$$

$$\Rightarrow \boxed{b_l = \frac{q}{\epsilon}s^l + a_l R^{2l+1}}$$

<u>normal component</u> $\vec{D}$ is continuous (since free surface charge $\sigma = 0$)

$$\vec{D} = \epsilon\vec{E}$$

$$\Rightarrow \epsilon E_r^{in} = E_r^{out}$$

$$-\epsilon\left.\frac{\partial\phi^{in}}{\partial r}\right|_R = -\left.\frac{\partial\phi^{out}}{\partial r}\right|_R$$

$$\Rightarrow \frac{(l+1)q}{R^2}\left(\frac{s}{R}\right)^l - l\epsilon a_l R^{l-1} = \frac{(l+1)b_l}{R^{l+2}}$$

$$qs^\ell - \frac{\ell}{\ell+1}\epsilon a_\ell R^{2\ell+1} = b_\ell$$

substitute in b_ℓ from previous boundary condition

$$qs^\ell - \frac{\ell}{\ell+1}\epsilon a_\ell R^{2\ell+1} = \frac{q}{\epsilon}s^\ell + a_\ell R^{2\ell+1}$$

$$qs^\ell\left[1 - \frac{1}{\epsilon}\right] = a_\ell R^{2\ell+1}\left[1 + \frac{\ell}{\ell+1}\epsilon\right]$$

$$\boxed{a_\ell = \frac{qs^\ell}{R^{2\ell+1}} \frac{\left[1 - \frac{1}{\epsilon}\right]}{\left[1 + \left(\frac{\ell}{\ell+1}\right)\epsilon\right]}}$$

$$b_\ell = \frac{q}{\epsilon}s^\ell + a_\ell R^{2\ell+1}$$

$$= \frac{q}{\epsilon}s^\ell + qs^\ell \frac{\left[1 - \frac{1}{\epsilon}\right]}{\left[1 + \left(\frac{\ell}{\ell+1}\right)\epsilon\right]}$$

$$b_\ell = \frac{qs^\ell}{\epsilon}\left\{1 + \frac{\epsilon - 1}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]$$

$$= \frac{qs^\ell}{\epsilon}\left[\frac{\epsilon\left(1 + \frac{\ell}{\ell+1}\right)}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]$$

$$\boxed{b_\ell = qs^\ell\left[\frac{1 + \left(\frac{\ell}{\ell+1}\right)}{1 + \left(\frac{\ell}{\ell+1}\right)\epsilon}\right]}$$

Check the result:

as $s \to 0$, should recover previous answer

for $s=0$, $a_\ell = b_\ell = 0$ for all $\ell \neq 0$

$$a_0 = \frac{q}{R}\left[1 - \frac{1}{\epsilon}\right]$$

$$b_0 = q$$

so $$\phi^{in}(\vec{r}) = \frac{q}{\epsilon r} + \frac{q}{R}\left[1 - \frac{1}{\epsilon}\right]$$

$$\vec{E}^{in} = -\vec{\nabla}\phi^{in} = \frac{q}{\epsilon r^2}\hat{r} \quad \text{as before}$$

$$\phi^{out}(\vec{r}) = \frac{q}{r}$$

$$\vec{E}^{out} = -\vec{\nabla}\phi^{out} = \frac{q}{r^2}\hat{r} \quad \text{as before}$$

Note: the constant that is the 2nd term in ϕ^{in} is just what is needed to make ϕ continuous at $r=R$

another check:

let $\epsilon \to \infty$ this models a conductor!

again one finds $a_\ell = b_\ell = 0$ for all $\ell \neq 0$

$a_0 = \frac{q}{R}$

$b_0 = q$

$\phi^{in}(\vec{r}) = \sum \frac{q}{\ell \epsilon r}\left(\frac{s}{r}\right)^\ell P_\ell + \frac{q}{R} \to \frac{q}{R}$ as $\epsilon \to \infty$

$\Rightarrow E^{in}(\vec{r}) = 0$ as ϕ^{in} is a constant.

$\phi^{out}(\vec{r}) = \frac{q}{r} \Rightarrow \vec{E}^{out} = \frac{q}{r^2}\hat{r}$

field outside is like point charge q at the origin, independent of where q is inside the sphere. This is the correct behavior of a conductor. The mobile charges in the conductor completely screen the q inside, and leave a uniform surface charge $\sigma_b = \frac{q}{4\pi R^2}$ on the surface.

Magnetostatics

Bar magnets - $\vec{j}=0$, $\vec{M}$ fixed and given (not a linear material)

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{H}=\frac{4\pi}{c}\vec{j}=0$$

$\vec{\nabla}\times\vec{H}=0 \Rightarrow \vec{H}=-\vec{\nabla}\Phi_M$ magnetic scalar potential

$$\vec{B}=\vec{H}+4\pi\vec{M}$$

$$\vec{\nabla}\cdot\vec{B}=\vec{\nabla}\cdot(\vec{H}+4\pi\vec{M})=0$$

$$\vec{\nabla}\cdot\vec{H}=-\nabla^2\Phi_M=-4\pi\,\vec{\nabla}\cdot\vec{M}$$

$$\nabla^2\Phi_M=4\pi\,\vec{\nabla}\cdot\vec{M}$$

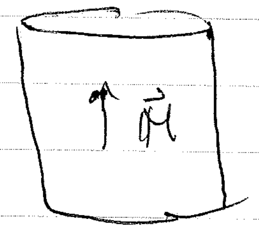
so $\rho_M\equiv-\vec{\nabla}\cdot\vec{M}$ looks like a magnetic "charge"

ρ_M is source for $\vec{H}$

Also at surfaces of material $\sigma_M=\hat{n}\cdot\vec{M}$ looks like surface charge

$$\vec{H}(\vec{r})=\int_V d^3r'\,\rho_M(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}+\oint_S da'\,\sigma_M(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

Field lines for $\vec{H}$ can start and end at sources and sinks given by ρ_M and σ_M



$\vec{M} = M\hat{z}$

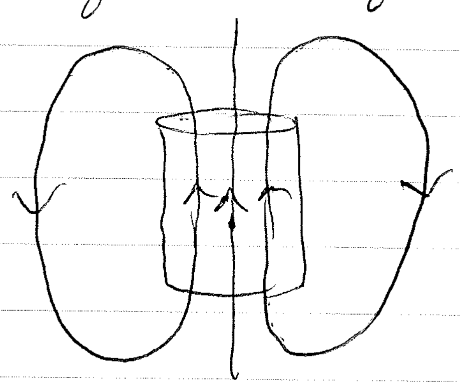
bound currents $\vec{J}_b = c\,\vec{\nabla}\times\vec{M} = 0$

$$\vec{K}_b = c\,\vec{M}\times\hat{n}$$

$$\vec{K}_b = \begin{cases} cM\hat{\phi} & \text{on side} \\ 0 & \text{on top \& bottom} \end{cases}$$

$\vec{K}_b$ is like solenoid current

field lines of $\vec{B}$ look like

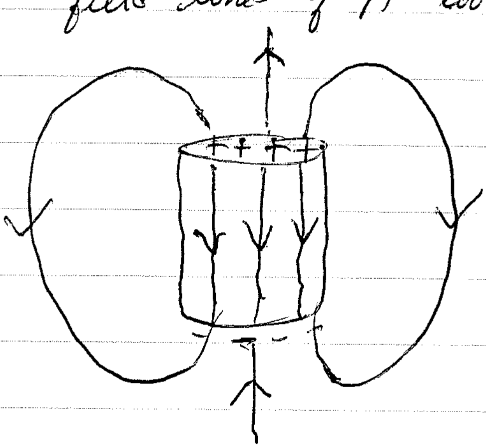


But $\vec{H}$ is determined as follows:

$$\rho_M = -\vec{\nabla}\cdot\vec{M} = 0$$

$$\sigma_M = \hat{n}\cdot\vec{M} = \begin{cases} M & \text{on top} \\ -M & \text{on bottom} \end{cases}$$

field lines of $\vec{H}$ look like parallel plate capacitor



field lines of $\vec{H}$ = field lines of $\vec{B}$ <u>outside</u> magnet, but they are very different inside the magnet!