Force and torque on electric dipoles

Localized charge distribution \( p(\vec{r}) \) with net charge \( \int d^3r \, p = 0 \)

Force on \( q \) in slowly varying electric field \( \vec{E} \) is

\[
\vec{F} = \int d^3r \, p(\vec{r}) \, \vec{E}(\vec{r})
\]

define \( \vec{r} = \vec{r}_0 + \vec{r}' \) where \( \vec{r}_0 \) is some fixed reference point in center of charge distribution \( p \), and \( \vec{r}' \) to distance relative to \( \vec{r}_0 \)

\[
\vec{F} = \int d^3r' \, p(\vec{r}') \, \vec{E}(\vec{r}_0 + \vec{r}')
\]

Since \( \vec{E} \) is slowly varying on length scale where \( p \neq 0 \), we expand

\[
\vec{F} = \int d^3r' \, p(\vec{r}') \left[ \vec{E}(\vec{r}_0) + (\vec{r}_0 \cdot \vec{\nabla}) \vec{E}(\vec{r}_0) \right] + \ldots
\]

\[
= \vec{E}(\vec{r}_0) \int d^3r' \, p(\vec{r}') + \left( \int d^3r' \, p(\vec{r}') \vec{r}' \cdot \vec{\nabla} \right) \vec{E}(\vec{r}_0)
\]

\[
= 0 + (\vec{p}, \vec{\nabla}) \vec{E}(\vec{r}_0)
\]

\[
\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} = \sum_{\alpha = 1}^{3} \varphi_\alpha \frac{\partial \vec{E}}{\partial r_\alpha}
\]

For \( \vec{E} = \text{constant} \), \( \vec{F} = 0 \)
Torque on $p$ is

$$\vec{T} = \int d^3 \tau \, j(\vec{r}) \vec{r} \times \vec{E}(\vec{r}) \approx \int d^3 \tau \, j(\vec{r}) \vec{r} \times [\vec{E}(\vec{r}) + \cdots]$$

to lowest order

$$\vec{T} = \vec{p} \times \vec{E}$$

Force and torque on magnetic dipoles

Localized magnetostatic current distribution $\vec{j}(\vec{r})$

$$\vec{F} = \frac{i}{c} \int d^3 \tau \, \vec{j} \times \vec{B}$$

Expand about center of current $\vec{r}_0$

$$\vec{B}(\vec{r}) = \vec{B}(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \vec{B}'(\vec{r}_0) + \cdots$$

$$\vec{F} = \frac{i}{c} \int d^3 \vec{r} \, \vec{j}(\vec{r}) \times \vec{B}(\vec{r}_0) + \frac{1}{c} \int d^3 \vec{r} \, \vec{j}(\vec{r}) \times (\vec{r} - \vec{r}_0) \cdot \vec{B}'(\vec{r}_0)$$

from discussion of magnetic dipole approx we had $\int d^3 \vec{r} \, \vec{j} = 0$

for magnetostatics where $\vec{V} \cdot \vec{j} = 0$, so 1st term vanishes.

The 2nd term can be written as

$$\vec{F}_d = \frac{\varepsilon_0 \mu_0}{c} \int d^3 \vec{r} \, \vec{j}_\mu \cdot \vec{r}_\nu \partial_\nu \vec{B}_\rho$$

we need the tensor $\frac{1}{c} \int d^3 \vec{r} \, \vec{j}_\mu \cdot \vec{r}_\nu = \frac{-i}{c} \int d^3 \vec{r} \, r_\mu' \vec{j}_\nu'$

$$= \frac{1}{2c} \int d^3 \vec{r} \, \left[ \vec{j}_\mu r_\nu' - r_\mu' \vec{j}_\nu \right]$$

$$= -\vec{\nabla} \times \vec{j}$$
\[ F_\alpha = \varepsilon_{\alpha \beta \gamma} \varepsilon_{\sigma \rho \delta} (-m_\sigma) \partial_\delta B_\beta \]

\[ = - (\delta_\alpha^0 \delta_\rho^1 - \delta_\rho^0 \delta_\alpha^1) m_\sigma \partial_1 B_0 \]

\[ = \ddot{\mathbf{M}} = \ddot{\nabla} (\ddot{\mathbf{M}} \cdot \ddot{\mathbf{B}}) - m_\sigma \ddot{\nabla} \cdot \ddot{\mathbf{B}} \]

\[ \ddot{\mathbf{F}} = \ddot{\nabla} (\ddot{\mathbf{M}} \cdot \ddot{\mathbf{B}}) \quad \text{as} \quad \ddot{\nabla} \cdot \ddot{\mathbf{B}} = 0 \]

**Torque on \( \ddot{\mathbf{J}} \):**

\[ \ddot{\mathbf{N}} = \frac{i}{c} \int d^3 r \ddot{\nabla} \times (\ddot{\mathbf{J}} \times \ddot{\mathbf{B}}) \quad \text{to lowest order,} \quad \ddot{\mathbf{B}} = \ddot{\mathbf{B}}(\ddot{\mathbf{r}}) \]

\[ = \frac{i}{c} \int d^3 r \left[ \ddot{\nabla} (\ddot{\mathbf{J}} \cdot \ddot{\mathbf{B}}) - \ddot{\mathbf{B}} (\ddot{\nabla} \cdot \ddot{\mathbf{J}}) \right] \]

2nd term \( = 0 \) as follows:

\[ \int d^3 r \ddot{\nabla} \ddot{\mathbf{J}} = \int d^3 r \ddot{\nabla} \ddot{\mathbf{J}} \cdot \ddot{\mathbf{B}} (\ddot{r}^2) \quad \text{as} \quad \ddot{\nabla} (\ddot{r}^2) = \ddot{r} \]

\[ = - \int d^3 r (\ddot{\nabla} \ddot{r}^2) \ddot{\mathbf{B}} (\ddot{r}^2) \quad \text{integrate by parts,} \]

\[ = 0 \quad \text{as} \quad \ddot{\nabla} \cdot \ddot{\mathbf{J}} = 0 \quad \text{in magnetostatics} \]

1st term involves \( \ddot{\mathbf{M}} \):

\[ \int d^3 r \ddot{\mathbf{J}} \ddot{\nabla} \ddot{\mathbf{M}} = - \int d^3 r \ddot{\nabla} \ddot{\mathbf{J}} \ddot{r} = \frac{1}{2} \int d^3 r \left[ \ddot{r} \ddot{\mathbf{B}} - \ddot{\mathbf{B}} \ddot{r} \right] \]

So

\[ \ddot{\mathbf{N}} = \frac{i}{2c} \int d^3 r \left[ \ddot{\nabla} (\ddot{r} \ddot{\mathbf{B}}) - \ddot{\mathbf{B}} (\ddot{r} \cdot \ddot{\mathbf{B}}) \right] \]
\[ \vec{N} = \frac{1}{2c} \int d^3r \left[ \vec{r} \times (\vec{r} \cdot \vec{B}) - \vec{r} (\vec{r} \cdot \vec{B}) \right] \times \vec{B} \]

\[ = \frac{1}{2c} \int d^3r \left( \vec{r} \times \frac{\vec{r} \cdot \vec{B}}{c} \right) \times \vec{B} \]

\[ \vec{N} = \vec{m} \times \vec{B} \]
Electrostatic energy of interaction

\[ E = \frac{1}{8\pi} \int d^3r \ E^2 \]

Suppose the charge density \( \rho \) that produces \( E \) can be broken into two pieces, \( \rho = \rho_1 + \rho_2 \) with \( E = E_1 + E_2 \) where \( \nabla \cdot E_1 = 4\pi \rho_1 \) and \( \nabla \cdot E_2 = 4\pi \rho_2 \). Then

\[ E = \frac{1}{8\pi} \int d^3r \left[ E_1^2 + E_2^2 + 2 E_1 \cdot E_2 \right] \]

"self-energy" "self-energy" "interaction" energy

of \( \rho_1 \) of \( \rho_2 \) of \( \rho_1 \) with \( \rho_2 \)

\[ E_{\text{int}} = \frac{1}{4\pi} \int d^3r \ E_1 \cdot E_2 \]

\[ = \int d^3r \ \rho_1 \phi_2 = \int d^3r \ \rho_2 \phi_1 \]

where \( \vec{E}_1 = -\nabla \phi_1 \), \( \vec{E}_2 = -\nabla \phi_2 \), by similar manipulations as earlier.

Integrals are over all space.

Apply to the interaction energy of a dipole in an external \( \vec{E} \) field

\[ E_{\text{int}} = \int d^3r \ \rho_1 \phi_2 \]

\( \phi \) potential of external \( \vec{E} \) field

\( \rho \) charge distribution of dipole
Assuming \( \phi_2 \) varies on length scale of \( \rho_1 \) then we can expand \( \phi_2(\vec{r}) = \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \nabla \phi_2(\vec{r}_0) \)

where \( \vec{r}_0 \) is the center of mass or any other convenient reference position within \( \rho_1 \)

\[
\text{\( \Sigma \text{Int} = \int d^3r \, \rho_1(\vec{r}) \left[ \phi_2(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \nabla \phi_2(\vec{r}_0) \right] \)}
\]

\[
= q \, \phi_2(\vec{r}_0) + \left[ \int d^3r \, \rho_1(\vec{r}) (\vec{r} - \vec{r}_0) \right] \cdot \nabla \phi_2(\vec{r}_0)
\]

\[
= q \, \phi_2(\vec{r}_0) + \vec{\rho} \cdot \vec{E}
\]

where \( q \) is total charge in \( \rho_1 \) and \( \vec{\rho} \) is dipole moment with respect to \( \vec{r}_0 \)

\( \vec{E} \) is external \( E \)-field

For a neutral charge distribution \( q = 0 \) and \( \vec{\rho} \) is independent of the origin about which it is computed, so

\[
\Sigma \text{Int} = -\vec{\rho} \cdot \vec{E}
\]

\( \Sigma \text{Int} \) is lowest when \( \vec{\rho} \parallel \vec{E} \)

\( \Rightarrow \) in thermal ensemble, dipoles tend to align parallel to an applied \( \vec{E} \)
Energy of magnetic dipole in external field

We had that the force on the dipole was

\[ \vec{F} = -\nabla (m \cdot \vec{B}) \]

If we regard this force as coming from the gradient of a potential energy \( U \) then \( \vec{F} = -\nabla U \Rightarrow \)

\[ U = -m \cdot \vec{B} \]

or equivalently, energy = work done to move dipole into position from \( \vec{0} \)

\[ W = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{l} = -\int_{\vec{r}_0}^{\vec{r}} \nabla (m \cdot \vec{B}) \cdot d\vec{l} = -m \cdot \vec{B}(\vec{r}) \]

This is the correct energy to use in cases where \( m \)

is due to intrinsic magnetic moments of atom or molecule—say from electron or nuclear spin. For a thermal ensemble, magnetic moments tend to align to \( \vec{B} \).

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we did if the the energy of an electric dipole in an electric field...
Magnetostatic energy of interaction

\[ \mathcal{E} = \frac{1}{8\pi} \int d^3r \ B^2 \]

Suppose current \( \mathcal{J} \) that produces \( B \) can be divided
\[ \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 \]
with \( B = B_1 + B_2 \), where \( \nabla \times B_1 = \frac{4\pi}{c} \mathcal{J}_1 \)
and \( \nabla \times B_2 = \frac{4\pi}{c} \mathcal{J}_2 \). Then

\[ \mathcal{E} = \frac{1}{8\pi} \int d^3r \left[ B_1^2 + B_2^2 + 2 B_1 \cdot B_2 \right] \]

self energy self energy interaction energy
of \( \mathcal{J}_1 \) of \( \mathcal{J}_2 \) of \( \mathcal{J}_1 \) with \( \mathcal{J}_2 \)

\[ \mathcal{E}_{\text{int}} = \frac{1}{4\pi} \int d^3r \ B_1 \cdot B_2 \]
\[ = \frac{1}{c} \int d^3r \ \mathcal{J}_1 \cdot \mathcal{A}_2 = \frac{1}{c} \int d^3r \ \mathcal{J}_2 \cdot \mathcal{A}_1 \]

where \( B_1 = \nabla \times \mathcal{A}_1 \), \( B_2 = \nabla \times \mathcal{A}_2 \), by similar manipulations as earlier

integrals are over all space

Apply to the interaction energy of a magnetic dipole in an external \( B \) field.

\[ \mathcal{E}_{\text{int}} = \frac{1}{2} \int d^3r \ \mathcal{J}_1 \cdot \mathcal{A}_2 \]
\( \mathcal{T} \) - vector potential of external \( B \) field
\( \mathcal{J} \) - current distribution of dipole
Assuming $A$ varies slowly on length scale of $\frac{c}{\omega}$, then expand $A_i(\vec{r}) = A_i(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \vec{\nabla} A_i(\vec{r}_0)$

$$E_{\text{int}} = \frac{1}{c} \int d^3r \ \vec{f}_i \cdot \vec{A}(\vec{r}_0)$$

$$+ \frac{1}{c} \int d^3r \ \sum_{i,j} \epsilon_{ikj} (\vec{r} - \vec{r}_0)_j \ \partial_j A_i(\vec{r}_0)$$

Dropped terms as we are interested in new monopoles.

From magnetostatic computation of magnetic dipole moment we had $\int d^3r \ \vec{f} = 0$

For magnetostatics

$\Rightarrow$ 1st term above vanishes. So does the piece of 2nd term $\left( \int d^3r \ \epsilon_{ikj} (\vec{r} - \vec{r}_0)_j \ \partial_j A_i(\vec{r}_0) \right)$

We are left with

$$E_{\text{int}} = \left[ \frac{1}{c} \int d^3r \ \epsilon_{ikj} R_j \right] \partial_j A_i(\vec{r}_0)$$

From computation of magnetic dipole approx we had

$$\int d^3r \ \epsilon_{ikj} R_j = - \int d^3r \ \delta_{ij} R_i$$

Recall:

$$m = \frac{1}{2c} \int d^3r \ \vec{r} \times \vec{f}$$

$$= \frac{1}{2} \int d^3r \left[ \epsilon_{ikj} r_j - \delta_{kj} r_i \right]$$

$$= \frac{1}{2} \epsilon_{kij} \int d^3r \ (\vec{f} \times \vec{r})_k$$

$$\Rightarrow \frac{1}{c} \int d^3r \ \delta_{ij} R_i = - \epsilon_{kij} \ m_b \ \text{mag dipole}$$
\[ \mathbf{E}_{\text{int}} = -m_k \varepsilon_{kij} \partial_j A_i = m_k \varepsilon_{kij} \partial_j A_i \]
\[ = m_o (\nabla \times \mathbf{A}) = m_o \mathbf{B} = \mathbf{E}_{\text{int}} \]

This is opposite in sign to what we found earlier!

Why the difference?

1. When we integrate the work done against the magnetostatic force to move \( m \) into position from infinity, we found the energy
\[ U = -m_o \mathbf{B} \]

2. When we compute the interaction energy from
\[ \mathbf{E}_{\text{int}} = \frac{1}{c^2} \int d^3r \mathbf{F}_1 \cdot \mathbf{A}_2 = \frac{1}{c^2} \int d^3r \int d^3r' \frac{\mathbf{F}_1(r) \cdot \mathbf{F}_2(r')}{|r-r'|} \]
we found the energy \( \mathbf{E}_{\text{int}} = +m_o \mathbf{B} \)

To see which is correct, let us consider computing the interaction energy (2) directly via method (1).
Consider two loops with currents $I_1$ and $I_2$.

What is the work done to move loop 2 in from infinity to its final position with respect to loop 1?

Magnetostatic force on loop 2 due to loop 1 is

$$ F = \frac{I_2}{c^2} \oint_{L_2} \vec{dl}_2 \times \vec{B}_1 \quad \text{Lorentz force} \quad \vec{B}_1 \text{ is magnetic field from loop 1} $$

$$ \vec{B}_1(r) = \frac{I_1}{c} \oint_{L_1} \vec{dl}_1 \times \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \quad \text{Biot-Savart law} $$

$$ F = \frac{I_1 I_2}{c^2} \int_{L_2} \oint_{L_1} \vec{dl}_2 \times \left( \vec{dl}_1 \times \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \right) $$

**Use triple product rule**

$$ \vec{dl}_2 \times \left[ \vec{dl}_1 \times \left( \vec{r}_2 - \vec{r}_1 \right) \right] = \vec{dl}_1 \left[ \vec{dl}_2 \cdot \left( \vec{r}_2 - \vec{r}_1 \right) \right] - \left( \vec{r}_2 - \vec{r}_1 \right) (\vec{dl}_1 \times \vec{dl}_2) $$

from the 1st term

$$ \oint_{L_2} \frac{\vec{dl}_2 \cdot \left( \vec{r}_2 - \vec{r}_1 \right)}{|\vec{r}_2 - \vec{r}_1|^3} = -\oint_{L_2} \frac{\vec{dl}_2 \cdot \nabla_2 \left( \frac{1}{|\vec{r}_2 - \vec{r}_1|} \right)}{|\vec{r}_2 - \vec{r}_1|^3} = 0 $$

as integral of gradient around closed loop always vanishes!
\[ F = -\frac{I_1 I_2}{c^2} \oint \oint \frac{\vec{d}l_1 \cdot \vec{d}l_2}{|r_2 - \vec{r}_1|^3} \cdot (\vec{r}_2 - \vec{r}_1) \]  \\
write \( \vec{r}_2 = \vec{R} + \delta \vec{r}_2 \) where \( \vec{R} \) is center of loop 2

use \( \frac{\vec{R} + \delta \vec{r}_2 - \vec{r}_1}{|\vec{R} + \delta \vec{r}_2 - \vec{r}_1|^3} = -\frac{\vec{V}_R}{|\vec{R} + \delta \vec{r}_2 - \vec{r}_1|} \)

\[ F = \frac{I_1 I_2}{c^2} \oint \oint \vec{d}l_1 \cdot \vec{d}l_2 \ \vec{V}_R \left( \frac{1}{|\vec{R} + \delta \vec{r}_2 - \vec{r}_1|} \right) \]

to move loop 2 we need to apply a force equal and opposite to the above magnetostatic force.

Therefore the work we do in moving loop 2 from infinity to its final position at \( \vec{R}_0 \) is

\[ W = -\int_{\infty}^{\vec{R}_0} F \cdot d\vec{R} = -\frac{I_1 I_2}{c^2} \oint \oint \frac{\vec{d}l_1 \cdot \vec{d}l_2}{|r_2 - \vec{r}_1|} \int_{\infty}^{\vec{R}_0} \vec{V}_R \left( \frac{1}{|\vec{R} + \delta \vec{r}_2 - \vec{r}_1|} \right) \]

\[ = -\frac{I_1 I_2}{c^2} \oint \oint \frac{\vec{d}l_1 \cdot \vec{d}l_2}{|r_2 - \vec{r}_1|} \]  \\
where \( \vec{r}_2 = \vec{R}_0 + \delta \vec{r}_2 \)

\[ = -\frac{1}{c^2} \int d^3r_1 \int d^3r_2 \ \frac{\vec{j}_1(\vec{r}_1) \cdot \vec{j}_2(\vec{r}_2)}{|r_2 - \vec{r}_1|} \]

\[ = -M_{12} \frac{I_1 I_2}{c^2} \]

\[ \vec{r} \] mutual inductance

\[ \text{note the minus sign!} \]

\[ \text{this is just the negative of the interaction energy!} \]
The minus sign we have here is the same minus sign we get when we found \( U = -\mathbf{m} \cdot \mathbf{a} \) by integrating the force on the magnetic dipole.

Why don't we get \[ + \frac{1}{c^2} \int d^3r_1 d^3r_2 \frac{\mathbf{f}_1(r_1) \cdot \mathbf{f}_2(r_2)}{|r_2 - r_1|} \]
with the plus sign we expect from \( E = \frac{1}{2} \int d^3r \ \mathbf{B}^2 \)?

*Answer:* we have left something out.

*Faraday's Law:* when we move loop 2, the magnetic flux through loop 2 changes. This \( \frac{d\Phi}{dt} \) creates an *emf* = \( \oint \mathbf{d} \mathbf{l} \cdot \mathbf{E} \) around the loop that would tend to change the current in the loop. If we are to keep the current fixed at constant \( I_2 \), then there must be a battery in the loop that does work to counter this induced emf (electromotive force).

Similarly, the flux through loop 1 is changing and a battery does work to keep \( I_1 \) constant. We need to add this work done by the battery to the mechanical work computed above.

\[
\text{emf induced in loop 1} \quad E_1 = \oint \mathbf{d} \mathbf{l}_1 \cdot \mathbf{E}_2 \\
\text{integrate once}
\]

\[
\text{emf induced in loop 2} \quad E_2 = \oint \mathbf{d} \mathbf{l}_2 \cdot \mathbf{E}_1 \\
\text{in direction of current}
\]

*Faraday* \[
E_1 = \frac{-d\Phi_1}{c \, dt} \quad \Phi_1 = \text{flux through loop 1}
\]

\[
E_2 = \frac{-d\Phi_2}{c \, dt} \quad \Phi_2 = \text{flux through loop 2}
\]
To keep the current constant, the batteries need to provide an emf that counters these Faraday induced emfs. The work done by the battery per unit time is therefore

\[
\frac{dW_{\text{battery}}}{dt} = -\varepsilon_1 I_1 - \varepsilon_2 I_2
\]

(check units: \(\varepsilon I\) is \([\text{length}] \cdot [\text{current}] \cdot [1/\text{s}]\)

\[
= [\text{length}] \cdot [\text{force}/\text{s}]\]

\[
= \text{energy/s}
\]

\[
\frac{dW_{\text{battery}}}{dt} = \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2
\]

Where \(t = 0\) loop 2 is at infinity,

\(t = T\) loop 2 is at final position,

\(I_1, I_2\) kept constant as loop moves

\[
W_{\text{battery}} = \int_{0}^{T} dt \left( \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2 \right)
\]

\[
W_{\text{battery}} = \frac{1}{c} \Phi_1 I_1 + \frac{1}{c} \Phi_2 I_2
\]

Where \(\Phi_1, \Phi_2\) are fluxes in final position, and are assumed to be 0 at infinity,

\[
\Phi_1 = c M_{12} I_2
\]

\[
\Phi_2 = c M_{21} I_1 = c M_{12} I_1\]

As \(M_{12} = M_{21}\)

\[
\Rightarrow W_{\text{battery}} = 2 M_{12} I_1 I_2
\]


add this to the mechanical work

\[ W_{\text{total}} = W_{\text{mech}} + W_{\text{battery}} = -M_{12} \mathbf{I}_1 \mathbf{I}_2 + 2M_{12} \mathbf{I}_1 \mathbf{I}_2 \]

\[ = M_{12} \mathbf{I}_1 \mathbf{I}_2 = \frac{1}{c^2} \int \frac{d^3 \mathbf{r}_1 \, d^3 \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\mathbf{f}_1(\mathbf{r}_1) \cdot \mathbf{f}_2(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \]

we get back the correct interaction energy!

**Conclusion:** The magneto-static interaction energy

\[ \frac{1}{c^2} \int \frac{d^3 \mathbf{r}_1 \, d^3 \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\mathbf{f}_1(\mathbf{r}_1) \cdot \mathbf{f}_2(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \]

includes the work done to maintain the currents stationary as the current distributions move.

When we computed the interaction energy of a current loop dipole \( \mathbf{m} \) and find

\[ E_{\text{int}} = +\mathbf{m} \cdot \mathbf{B} \]

this includes the energy needed to maintain the constant current producing the constant \( \mathbf{m} \)

When we integrated the force on the dipole to find the potential energy

\[ U = -\mathbf{m} \cdot \mathbf{B} \]

this did not include the energy needed to maintain the constant current that creates \( \mathbf{m} \). This is the correct energy expression to use when \( \mathbf{m} \) comes from intrinsic magnetic moments (due to particles intrinsic spin, which cannot be viewed as arising from a current loop).
